

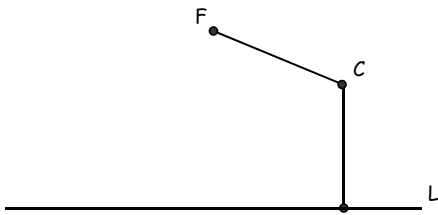
The Terrible Parable of the Parabola

MA 341 - Topics in Geometry
Lecture 28



Parabola

Given a line L and a point F not on L , the locus of points that are equidistant from L and from F is called a parabola

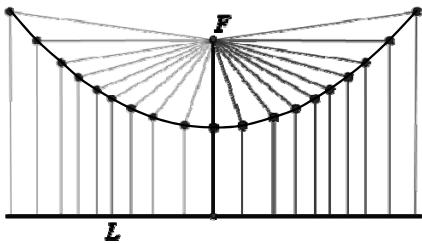


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Parabola



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Parabola

F = focus

L = directrix

Line perpendicular to L through F = axis

Midpoint of LF = vertex

Chord passing through focus = focal chord

Chord passing through focus perpendicular to axis = latus rectum (right chord)

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Parabola

Lemma 1: Latus rectum = 2 x distance from focus to vertex.

Lemma 2: The parabola having directrix $y = -a$ and focus $(0,a)$ has equation $x^2 = 4ay$.

Lemma 3: The parabola having axis parallel to the y -axis and vertex (h,k) has the equation $(x-h)^2 = 4a(y-k)$.

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Reflection Properties

Start with the parabola $y^2 = 4ax$

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Reflection Properties

Reflects where?

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Reflection Properties

Angle of incidence = angle of reflection

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Reflection Properties

Reflects off tangent line!

$\alpha = \beta$

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Reflection Properties

We need to find α and β .

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Reflection Properties

Since ray is horizontal, α is same as the angle the green line makes with the x-axis.

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Reflection Properties

Slope = tangent of angle line makes with the x-axis

$\tan \alpha = m_{\text{tangent}}$

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Reflection Properties

$P=(p,q)$

$\tan(\alpha) = m_{\text{tangent}}$

$m_{\text{tangent}} = \frac{dy}{dx}$

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Reflection Properties

$P=(p,q)$

$m_{\text{tangent}} = \frac{dy}{dx}$

$2y \frac{dy}{dx} = 4a$

$m_{\text{tangent}} = \frac{2a}{y} = \frac{2a}{q} = \frac{2a}{2\sqrt{ap}} = \sqrt{\frac{a}{p}}$

$\tan \alpha = \sqrt{\frac{a}{p}}$

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Reflection Properties

$P=(p,q)$

$\beta = \alpha$ so
 $\tan \beta = \tan \alpha$, so how do
 we find the slope of
 the reflected line?
 Since $\beta = \alpha$ angle that
 reflected line makes
 with x-axis is 2α so

$m = \tan(2\alpha)$

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Reflection Properties

$P=(p,q)$

$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

$$= \frac{2\sqrt{\frac{a}{p}}}{1-\frac{a}{p}} = \frac{2\sqrt{ap}}{p-a}$$

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Reflection Properties

Equation of line:
 $y - q = \frac{2\sqrt{ap}}{p-a}(x - p)$

Where does this intersect the x-axis?

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Reflection Properties

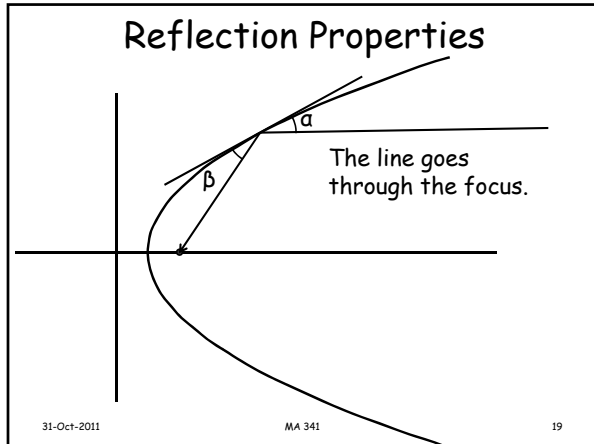
$$0 - q = \frac{2\sqrt{ap}}{p-a}(x - p)$$

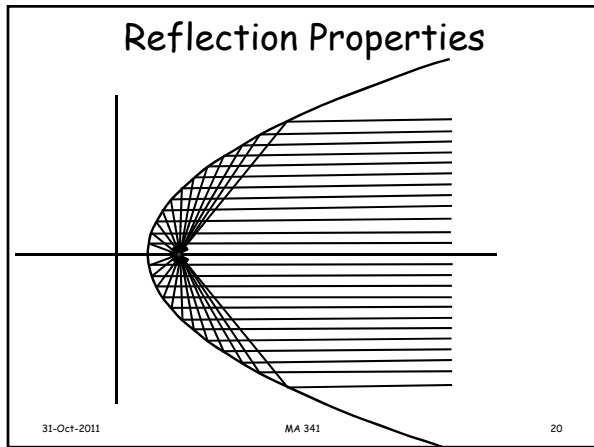
$$q(a - p) = 2\sqrt{ap}(x - p)$$

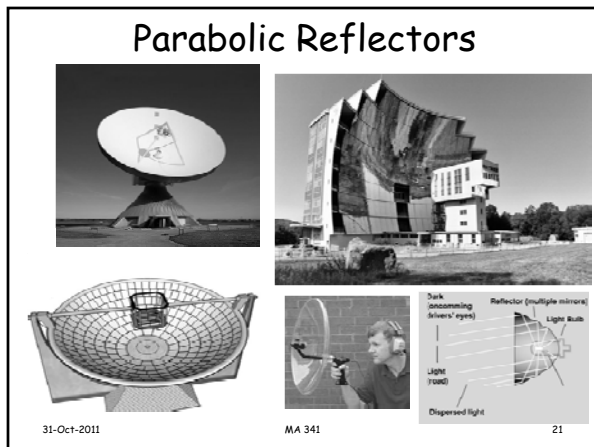
$$x = \frac{q(a - p)}{2\sqrt{ap}} + p$$

$$x = \frac{2\sqrt{ap}(a - p)}{2\sqrt{ap}} + p = a$$

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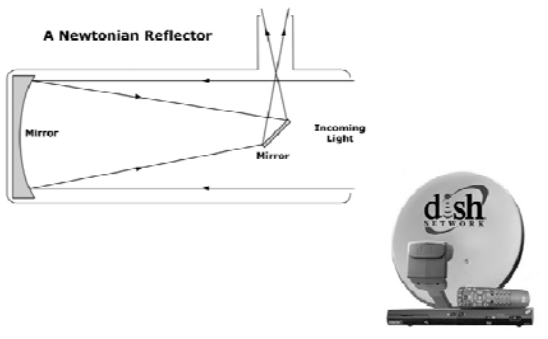






Reflecting Telescope

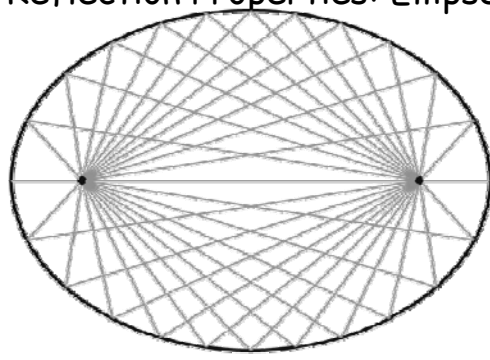
A Newtonian Reflector



The diagram shows a Newtonian reflector telescope with a primary mirror at the back and a secondary mirror at the front. Incoming light enters from the right, reflects off the primary mirror, then off the secondary mirror, and exits through a side opening. Below the diagram is a photograph of a dish antenna with the 'dish NETWORK' logo.

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Reflection Properties: Ellipse



The diagram shows an ellipse with two foci. Numerous lines radiate from each focus to the perimeter of the ellipse, illustrating the reflection property of an ellipse.

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Reflection Properties: Ellipse

Lookup LITHOTRIPTER
WHISPERING CHAMBER

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Hyperbola

A line directed at one focus is reflected to the other

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Cassegrain Telescope

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Rotation of Axes

General quadratic equation takes the form:
 $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

How can we tell what conic it represents?

If $B = 0$, then completing the square will tell us.

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Translation of Axes

What conic section is represented by:
 $9x^2 + 16y^2 - 18x + 64y - 71 = 0$

$$9(x^2 - 2x) + 16(y^2 + 4y) - 71 = 0$$

$$9(x^2 - 2x + 1) + 16(y^2 + 4y + 4) = 71 + 9 + 16(4)$$

$$9(x - 1)^2 + 16(y + 2)^2 = 144$$

$$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$$

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Rotation of Axes

If $B \neq 0$, then a rotation is required to remove the xy term.

The standard rotation of axes is

$$x = x' \cos a - y' \sin a \quad \text{and} \quad y = x' \sin a + y' \cos a$$

The equation becomes:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$A(x' \cos a - y' \sin a)^2 + B(x' \cos a - y' \sin a)(x' \sin a + y' \cos a) + C(x' \sin a + y' \cos a)^2 + D(x' \cos a - y' \sin a) + E(x' \sin a + y' \cos a) + F = 0$$

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Rotation of Axes

$$A((x')^2 \cos^2 a - 2x'y' \cos a \sin a + (y')^2 \sin^2 a) + B((x')^2 \cos a \sin a - x'y' \sin^2 a + x'y' \cos^2 a + (y')^2 \cos a \sin a) + C((x')^2 \sin^2 a + 2x'y' \sin a \cos a + (y')^2 \cos^2 a) + D(x' \cos a - y' \sin a) + E(x' \sin a + y' \cos a) + F = 0$$

$$(A \cos^2 a + B \cos a \sin a + C \sin^2 a)(x')^2 + (B(\cos^2 a - \sin^2 a) + (2C - 2A)\sin a \cos a)x'y' + (C \cos^2 a - B \cos a \sin a + A \sin^2 a)(y')^2 + (D \cos a + E \sin a)x' + (E \cos a - D \sin a)y' + F = 0$$

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Rotation of Axes

In order to have no "cross term" we need

$$(B(\cos^2 \alpha - \sin^2 \alpha) + (2C - 2A)\sin \alpha \cos \alpha) = 0$$

$$B \cos 2\alpha + (C - A)\sin 2\alpha = 0$$

$$(A - C)\sin 2\alpha = B \cos 2\alpha$$

$$\tan 2\alpha = \frac{B}{A-C} \quad \text{or} \quad \cot 2\alpha = \frac{A-C}{B}$$

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Rotation of Axes

What conic section is represented by:

$$x^2 + 4xy + 4y^2 - 30x - 90y - 450 = 0 ?$$

Angle of rotation: $\tan 2\alpha = B/(A-C) = -4/3$

Then $\cos 2\alpha = -3/5$

Then

$$\cos \alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}} = \sqrt{\frac{1 + (-3/5)}{2}} = \frac{1}{\sqrt{5}}$$

$$\sin \alpha = \sqrt{\frac{1 - \cos 2\alpha}{2}} = \sqrt{\frac{1 - (-3/5)}{2}} = \frac{2}{\sqrt{5}}$$

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Rotation of Axes

Now, from before we have:

$$A' = A \cos^2 \alpha + B \cos \alpha \sin \alpha + C \sin^2 \alpha = 5$$

$$B' = B \cos 2\alpha + (C - A)\sin 2\alpha = 0$$

$$C' = C \cos^2 \alpha - B \cos \alpha \sin \alpha + A \sin^2 \alpha = 0$$

$$D' = D \cos \alpha + E \sin \alpha = -42\sqrt{5}$$

$$E' = E \cos \alpha - D \sin \alpha = -6\sqrt{5}$$

$$F' = 450$$

So the equation after rotation becomes:

$$5(x')^2 - 42\sqrt{5}x' - 6\sqrt{5}y' + 450 = 0$$

$$\left(x' - \frac{21}{\sqrt{5}}\right)^2 = \frac{6}{\sqrt{5}}\left(y' - \frac{3\sqrt{5}}{10}\right)$$

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Classification of Quadratics

For the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Define the discriminant to be $B^2 - 4AC$

Theorem:

- 1) if $B^2 - 4AC > 0$, the graph is a hyperbola;
- 2) if $B^2 - 4AC = 0$, the graph is a parabola;
- 3) if $B^2 - 4AC < 0$, the graph is an ellipse,

except for degenerate cases

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