

Chapter 3

Euclidean Constructions

The idea of constructions comes from a need to create certain objects in our proofs. A construction is, in some sense, a physical substantiation of the abstract.

In Greek times, geometric constructions of figures and lengths were restricted to the use of only a straightedge and compass (or in Plato's case, a compass only). No markings could be placed on the straightedge to be used to make measurements. While there is only one type of straightedge - a ruler without markings - there are two types of compasses. The Greeks, from Plato, use dividers, or what we now call a collapsing compass. This compass loses the radius once the instrument is moved. This would mean that you could not construct a circle and then move the compass and construct a circle of the same radius. Furthermore, this compass could not be used even to mark off distances by setting it and then *walking* it along.

Because of the prominent place Greek geometric constructions held in Euclid's Elements, these constructions are sometimes also known as Euclidean constructions. Such constructions lay at the heart of the three classical geometric problems (1) squaring the circle (also known as *quadrature of the circle*), (2) duplicating the cube, and (3) trisecting an arbitrary angle. The Greeks were unable to solve these problems. It was not until the eighteenth and nineteenth century that the problems were proved to be actually impossible under the limitations imposed.

The constructions are all related to Euclid's first three axioms:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To draw a circle with any center and radius.

3.1 Different Compasses?

The straightedge is thought of as having no length markings because none of these postulates provides us with the ability to measure lengths. Thus you may use the straightedge to connect a pair of given points or extend a given line segment arbitrarily in a straight line. However, you cannot use the straightedge to extend a line segment 3 inches. Since the straightedge is the same, this does not bother us. However, since the Euclidean compass is different from the modern, or fixed, compass, we do have some concerns. Can we make the same constructions? Are the rules the same for both compasses? Well, we can.

Theorem 3.1 (The Compass Equivalence Theorem) *A circle $C(B, r)$ can be congruently copied (using only a straightedge and collapsing compass) so that a given point A serves as the center of the copy.*

Proof: Consider the given circle and a point A as shown to the right.

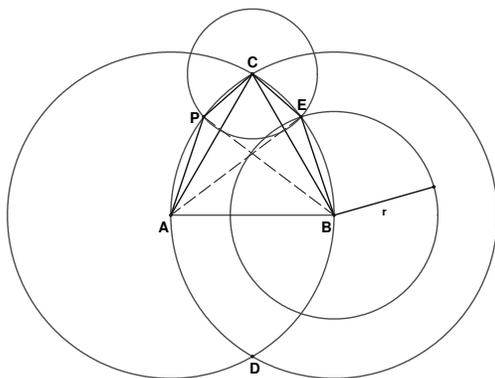
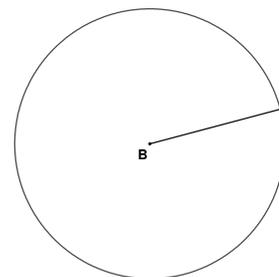
We need to construct a circle of radius r centered at A , using only a straightedge and collapsing compass. First construct the circle centered at A with radius AB and the circle centered at B with radius BA . These circles intersect in two points. Call the points C and D . $C(B, r)$ and $C(A, AB)$ intersect in a point E . Consider the circle $C(C, CE)$. It will intersect $C(B, BA)$ in a point P . We claim that $|AP| = r$.

$\triangle PCB \cong \triangle ECA$ by SSS. Thus, $\angle PCB \cong \angle ECA$, and

$$\angle PCB - \angle ACB = \angle ECA - \angle ACB.$$

Therefore, $\angle PCA \cong \angle ECB$.

Now, $CP \cong CE$ and $AC \cong BC$. Thus, by SAS, $\triangle APC \cong \triangle BEC$, which implies that $|AP| = |BE| = r$.



3.2 Euclidean Constructions

Now, we can discuss the constructions using the Euclidean axioms, but our compass.

Construction 1: We can construct the perpendicular bisector of a line segment.

Construction 2: We can construct a perpendicular line from a given point to a line not containing that point.

Construction 3: We can construct a line perpendicular to a given line at a given point P on ℓ .

Construction 4: We can bisect an arbitrary angle.

Construction 5: We can copy a given angle to a given ray with a given vertex.

Construction 6: Given a line ℓ and a point P not on ℓ , we can construct a line through P parallel to the given line.

Construction 7: We can construct an equilateral triangle.

Construction 8: We can construct a square.

Construction 9: We can construct a regular hexagon.

These are the usual constructions. We have many more. In fact, the process of constructions can (and does) form an algebraic system. We need to know how to add, subtract, multiply and divide segments. First, we will recall a construction that you may or may not recall from your high school geometry course.

Let us start with two points. We will normalize our measurement of length by defining the distance between those two points to be equal to one unit. We say that a length a is a *constructible length* if there are two constructible points P and Q so that $|PQ| = a$.

Construction 10: We can partition a line segment into three (or more) congruent pieces. In other words, if a is constructible then so is a/n for any positive integer, n .

It is quite simple, so let us illustrate it with $n = 3$. Let AB denote the segment that we wish to divide into three congruent sub-segments. Choose a point, C , not on the line AB . We would prefer that you choose C so that the angle $\angle CAB$ is an acute or right angle, but this is not necessary. Choose a point, D_1 , on the ray AC . This could be the point C , but need not be. Normally, we would choose the segment, AD_1 , to have unit length, but again this is not necessary. Copy AD_1 twice more on the ray AC , to get two more segments, D_1D_2 and D_2D_3 , each of which is congruent to AD_1 . Thus the final segment has the property that $|AD_3| = 3|AD_1|$. Now, construct the segment D_3B . Construct the lines through D_2 and D_1 parallel to the line BD_3 . Each will intersect AB in a point. Call these points P and Q . Then, $AP \cong PQ \cong QB$ and you have trisected the segment.

Construction 11: If a and b are constructible, then so are $a + b$ and $a - b$.

This one should be obvious, but let's go over the proof in order to see how we need to prove these results.

PROOF: Let A and A' be points so that $|AA'| = a$ and B and B' be points so that $|BB'| = b$. Draw line AA' and construct the circle centered at A' of radius b . This circle will intersect the line AA' in two points, C and C' . Let C be on the opposite side of A from A' , then $|AC| = a + b$ and $|AC'| = a - b$. ■

Construction 12: If a and b are constructible, then so is ab .

PROOF: We will use similar triangles. Suppose that AB has length a and CD has length b . How do we construct a segment that has length ab ?

On a constructible line PQ containing point O , find points I and E so that $|OI| = 1$ and $|OE| = a$. Now choose a point K on the perpendicular to PQ at I so that $|KI| = b$. Construct the line OK and the perpendicular to PQ at E . These lines intersect in a point F . Then the triangles $\triangle OIK$ and $\triangle OEF$ are similar and

$$\frac{b}{1} = \frac{|EF|}{c},$$

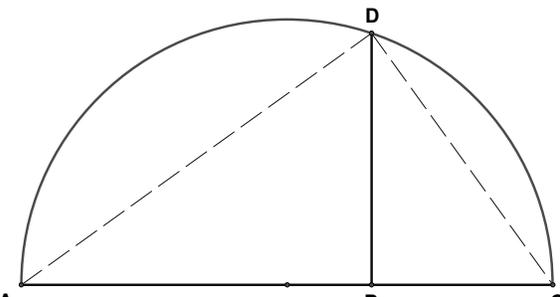
and we are done. ■

Construction 13: If a and b are constructible, then so is a/b .

We have the set of constructible numbers. How big is it? Clearly, we can perform addition, subtraction, multiplication and division. This means that we can construct all integers and rational numbers. Can we construct more? We suspect so. We can surely construct $\sqrt{2}$, $\sqrt{3}$ and others. Given a number a , that is constructible, can we construct \sqrt{a} .

Construction 14: If a is constructible, then \sqrt{a} is constructible.

First, assume that AB has length a and BC has length 1. We construct a circle with AC as a diameter and let the line perpendicular to AC passing through B intersect the circle at the point D . Then, $\triangle ADB \sim \triangle DCB$ and



$$\begin{aligned}\frac{|AB|}{|BD|} &= \frac{|DB|}{|BC|} \\ \frac{a}{|BD|} &= \frac{|BD|}{1} \\ a &= |BD|^2 \\ |BD| &= \sqrt{a}\end{aligned}$$

3.3 The Algebra of Constructible Numbers

The theory of constructible numbers is pretty well covered by the next two theorems.

Theorem 3.2 *The use of a straightedge alone can never yield segments of numbers outside the original number field.*

If a is an element of the number field so that \sqrt{a} is not an element of our number field, then our construction technique can extend this number field by making \sqrt{a} constructible. Then we can construct all numbers of the form $b + c\sqrt{a}$, where b and c are in the number field. Now, the set of all numbers of this form also forms a field, called an *extension field* of the original field of numbers.

Theorem 3.3 *A real number is constructible if and only if it is of the form $b + c\sqrt{a}$, where each of a , b , and c are either rational numbers or are constructed from repeated extraction of square roots of rational numbers.*

The best way to state this is that this number must lie in an extension field of the rational numbers of degree $2n$.

Construction 15: Construct a regular pentagon with a given side. (We defer this until the next section.)

Construction 16: Construct a regular n -gon with a given side. Is this possible for all n ? For which n is it possible?

Gauss proved the following result:

Theorem 3.4 (Gauss) *A regular polygon can be inscribed in a circle by means of a straightedge and compass alone if and only if the number of sides, n , can be expressed as $n = 2^k \cdot p_1 p_2 \dots p_m$, for a nonnegative integer k and each p_i a distinct prime of the form $2^{2^r} + 1$, for $r > 0$.*

Some of the regular polygons that are constructible, according to this theorem, are those with 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, or 24 sides. Note that the theorem does not tell us how to do the construction, only that it can or cannot be done.

3.4 The Regular Pentagon

Rather than simple directions for constructing a regular pentagon, this is an opportunity to tie together two areas of mathematics and make some *connections*.

Consider the regular pentagon $ABCDE$ inscribed in a circle of radius one centered at O , as above. Let the chord BE intersect the radius OA at the point F . Then we have

$$|OF| = \cos \angle BOA = \cos 72^\circ.$$

Therefore, if we can construct the number, $\cos 72^\circ$, then we can construct F , and hence the regular pentagon. Once we have constructed F , we would then construct the perpendicular to OA at F . This will intersect the circle in two points, giving us three of the vertices of the regular pentagon. Using circles we get the other two vertices.

How will we construct $\cos 72^\circ$? We will appeal to the geometry of complex numbers. Recall that from DeMoivre's theorem we can represent any complex number as:

$$a + bi = r \cdot e^{i\theta} = r(\cos \theta + i \sin \theta),$$

where θ is the principal angle between the positive real axis and the given complex number. Here, since we are on the circle of radius 1, we have that any number on the unit circle can be represented by $e^{i\theta} = \cos \theta + i \sin \theta$. Set $\theta = \frac{2\pi}{5} = 72^\circ$. For simplicity, let's write $\omega = e^{i(2\pi/5)}$. Now, note that:

$$\left(e^{i(2\pi/5)}\right)^5 = e^{2\pi i} = 1$$

so that $\omega^5 - 1 = 0$. Factoring, we have

$$(\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0.$$

Since $\omega \neq 1$, we have to have that the second factor must be zero. Divide both sides of this equation by ω^2 .

$$\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0.$$

Set $x = \omega + \omega^{-1}$. Now,

$$\begin{aligned} x &= \omega + \omega^{-1} \\ &= e^{2\pi i/5} + e^{-2\pi i/5} \\ &= \cos(2\pi/5) + i \sin(2\pi/5) + \cos(-2\pi/5) + i \sin(-2\pi/5) \\ &= \cos(2\pi/5) + i \sin(2\pi/5) + \cos(2\pi/5) - i \sin(2\pi/5) \\ &= 2 \cos(2\pi/5) \end{aligned}$$

Note that

$$x^2 = (\omega + \omega^{-1})^2 = \omega^2 + 2 + \omega^{-2}$$

Thus,

$$x^2 + x = \omega^2 + 2 + \omega^{-2} + \omega + \omega^{-1} = (\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2}) + 1$$

It then follows from above that

$$x^2 + x - 1 = 0.$$

From the quadratic formula we get

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Since x must be positive, we get

$$2 \cos(2\pi/5) = x = \frac{-1 + \sqrt{5}}{2}$$

$$\cos(2\pi/5) = \frac{-1 + \sqrt{5}}{4}$$

This is clearly constructible, so the regular pentagon is constructible.

The following construction is due to H.W. Richmond.

Theorem 3.5 *Let $|OA|$ have length 1, and construct the circle with center O and radius 1. Let E be the intersection of this circle with the perpendicular to OA at O . Let B be the midpoint of OE , and let the angle bisector of $\angle ABO$ intersect OA at C . Then*

$$|OC| = \frac{-1 + \sqrt{5}}{4}.$$

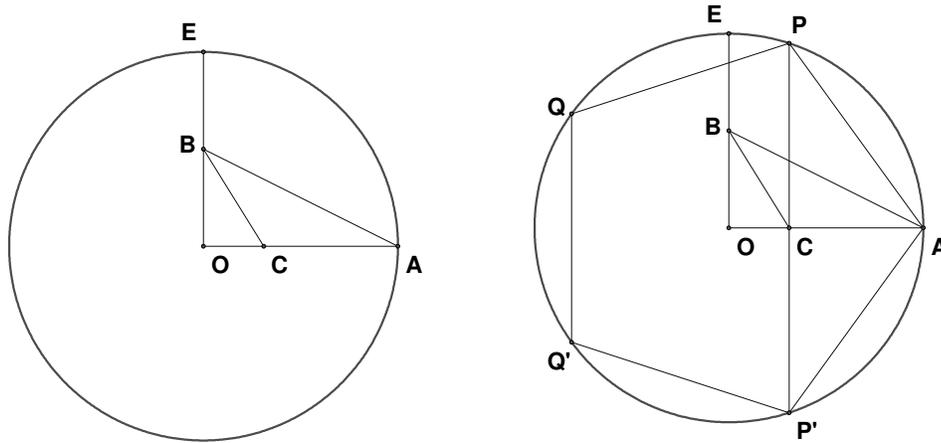


Figure 3.1: Richmond's construction of the regular pentagon

PROOF: Let $\angle CBO = \alpha$ and let $\tau = \tan \alpha$. Then

$$\tau = \tan \alpha = \frac{|OC|}{|OB|} = \frac{|OC|}{1/2} = 2|OC|.$$

Also,

$$\tan(2\alpha) = \frac{|OA|}{|OB|} = \frac{1}{1/2} = 2.$$

Recall the double angle formula for tangents:

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

So,

$$2 = \frac{2\tau}{1 - \tau^2}$$

$$1 - \tau^2 = \tau$$

$$\tau^2 + \tau - 1 = 0$$

$$\tau = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $|OC| > 0$, we then get that $|OC| = \frac{-1+\sqrt{5}}{4}$, as we needed.

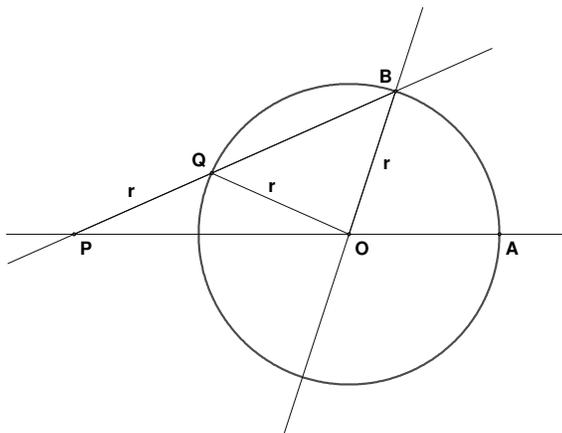
To finish the construction of the regular pentagon, just find the perpendicular to OA at C which will intersect the circle at points P and P' . Use the length AP to find the points Q and Q' to get the regular pentagon. ■

3.5 Trisecting an Arbitrary Angle

The usual proof that you cannot trisect an arbitrary angle involves showing that you have to construct a particular number which is not constructible — . The Greeks did not just work with a straightedge and compass — if they could solve a construction using only these tools, then they were considered "better solutions." There were several different techniques for trisecting an angle — it is just that none of them used only a compass and straightedge. To see how restrictive a straightedge is, just consider the following result.

Theorem 3.6 (Archimedes) *If we are in possession of a compass and a straightedge that is notched in two places, then it is possible to trisect an arbitrary angle.*

All I need to do is to make two little marks on my straightedge and I can solve the problem! How?



PROOF: Let the arbitrary angle be given by the intersection of lines ℓ_1 and ℓ_2 , which intersect at O . This proof shows how to trisect the acute angle in the intersection.

Let r denote the distance between the two notches on the straightedge. Draw a circle centered at O of radius r . It will intersect ℓ_1 and ℓ_2 at A and B so that the acute angle $\angle AOB$ is the angle to be trisected.

Now, put one notch of the straightedge on the line ℓ_2 and the other notch on the circle, and move the straightedge until it goes through B . So we have a point P on ℓ_2 and Q on the circle so that $|PQ| = r$.

Let $\angle QPO = \alpha$. Since $\triangle QPO$ is isosceles, $\angle QOP = \alpha$. Thus, $\angle OQB = 2\alpha$, since it is an exterior angle to $\triangle QOP$. Since $\triangle QOB$ is isosceles, $\angle QBO = 2\alpha$ and $\angle BOQ = 180^\circ - 4\alpha$. Now, we also have that

$$\angle POQ + \angle QOB + \angle AOB = 180,$$

so substituting we get

$$\begin{aligned}\angle AOB &= 180^\circ - (\angle POQ + \angle QOB) \\ &= 180^\circ - \alpha - (180^\circ - 4\alpha) \\ &= 3\alpha.\end{aligned}$$

So, $\angle QOP = \frac{1}{3}\angle AOB$, as desired. ■