Chapter 6

Neutral Geometry

How much of what we know about geometry depends on the "parallel postulate" regardless of which formulation we take? How much of what we take for granted really is independent of Euclid's Fifth postulate? We will take a look at that in this chapter. We will put off choosing any particular choice about parallels until we have to make that choice.

6.1 Axiom System

We will follow a system of axioms that is less restrictive than Hilbert's system. We will follow very closely to the SMSG axiom system, but we want to remove redundancy and be as rigorous as we can. We follow the system of axioms found in *Foundations of Geometry* by Venema. It is a good middle of the road axiom system that will allow us to study geometry without bogging us down too much.

6.1.1 Undefined Terms

We will accept five undefined terms for now — *point, line, distance, half-plane,* and *angle measure*. Later we will add the undefined term *area* to our list. Our axiom system tells us how these undefined terms relate to one another.

6.1.2 Existence and Incidence

We need to know that points exist. Now, we normally just blithely forge ahead and really never think about it, but we use this implicitly quite a bit. We need an axiom to insure that we actually have points when we need them.

Axiom 1 (The Existence Axiom) *The collection of all points forms a nonempty set. There is more than one point in that set.*

We call the set of all points the *plane* and will denote it by $\mathbb{P}$.

Now, how are lines and points related in the plane? We expect lines to have points on them and we expect for points to determine lines. We need an axiom to deal with this.

Axiom 2 (The Incidence Axiom) *Every line is a set of points. For every pair of distinct points A and B there is exactly one line ℓ such that $A \in \ell$ and $B \in \ell$.*

Now we make a few definitions.

Definition 6.1 *A point P is said to* lie *on line ℓ if $P \in \ell$. A point Q is said to be* external *to ℓ if $Q \notin \ell$. Two lines ℓ and m are said to be* parallel *if $\ell \cap m = \emptyset$.*

Note that we have excluded a line being parallel to itself!

Theorem 6.1 *If ℓ and m are two distinct, nonparallel lines, then there is exactly one point P so that P lies on both ℓ and m.*

6.1.3 Distance

Axiom 3 (The Ruler Axiom) *For every pair of points P and Q there exists a real number $d(P,Q)$, called the distance from P to Q. For each line ℓ there is a one-to-one correspondence from ℓ to $\mathbb{R}$ such that if P and Q are points on the line that correspond to the real numbers x and y, then $d(P,Q) = |x - y|$.*

Definition 6.2 *Let A, B, and C be three distinct points. The point C is* between *A and B, written $A * C * B$, if $C \in \overleftrightarrow{AB}$ and $d(A,C) + d(C,B) = d(A,B)$.*

Definition 6.3 *Define the segment AB by*

$$AB = \{A, B\} \cup \{P \mid A * P * B\}$$

and the ray $\overrightarrow{AB}$ by

$$\overrightarrow{AB} = AB \cup \{P \mid A * B * P\}.$$

Definition 6.4 *The* length *of segment AB is $d(A,B)$. Two segments AB and CD are* congruent*, written $AB \cong CD$, if they have the same length.*

Theorem 6.2 *If P and Q are any points,then*

i) $d(P,Q) = d(Q,P)$.

ii) $d(P,Q) \geq 0$.

iii) $d(P,Q) = 0$ *if and only if* $P = Q$.

6.1.4 Plane Separation

This axiom will tell us that the plane is two-dimensional in the sense that a line separates it into two disjoint sets, and allows us to define angle, interior of an angle and triangle.

Axiom 4 (The Plane Separation Axiom) *For every line ℓ the points that do not lie on ℓ form two disjoint, nonempty sets H_1 and H_2, called half-planes bounded by ℓ, such that the following conditions are satisfied:*

i) Each of H_1 and H_2 is convex. [1]

[1] A set is *convex* if for every pair of points in the set the line segment joining those two points is also in the set.

ii) If $P \in H_1$ and $Q \in H_2$ then $PQ \cap \ell \neq \emptyset$.

In terms of more familiar notation, that will occur later in the class, let's sum up what this axiom tells us about H_1 and H_2.

i) $H_1 \cup H_2 = \mathbb{P} \setminus \ell$.

ii) $H_1 \cap H_2 = \emptyset$.

iii) $H_1 \neq \emptyset$ and $H_2 \neq \emptyset$.

iv) If $A, B \in H_1$ then $AB \subset H_1$ and $AB \cap \ell = \emptyset$.

v) If $A, B \in H_2$ then $AB \subset H_2$ and $AB \cap \ell = \emptyset$.

vi) If $A \in H_1$ and $B \in H_2$, then $AB \cap \ell \neq \emptyset$.

If ℓ is a line and $A \notin \ell$ then we will use the notation H_A to denote the half-plane bounded by ℓ containing A.

Definition 6.5 *Let ℓ be a line and let A and B be two external points. We say that A and B are* on the same side of ℓ *if $AB \cap \ell = \emptyset$. We say that A and B are* on opposite sides of ℓ *if $AB \cap \ell \neq \emptyset$.*

Definition 6.6 *Two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ having the same endpoints are* opposite *rays if the two rays are unequal but $\overleftrightarrow{AB} = \overleftrightarrow{AC}$. Otherwise, they are* nonopposite.

Note that another way to state this is that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays if $B * A * C$.
Now we are ready to define *angle.*

Definition 6.7 *An* angle *is the union of two nonopposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ sharing the same endpoint. This angle is denoted by either $\angle BAC$ or $\angle CAB$. The point A is called the* vertex *of the angle and the rays are called the* sides *of the angle.*

Note that our definition of angle demands that the sides be "nonopposite" rays, so that we do not have a "straight angle". One reason for doing it this way is to allow us to make the following definition.

Definition 6.8 *Let $\angle BAC$ be given. The* interior *of angle $\angle BAC$ is defined as follows. If $\overrightarrow{AB} \neq \overrightarrow{AC}$ then the interior of the angle is defined to be the intersection of the half plane H_B determined by B and $\overleftrightarrow{AC}$ and the half plane H_A determined by C and $\overleftrightarrow{AB}$. If $\overrightarrow{AB} = \overrightarrow{AC}$ then the interior is defined to be the empty set.*

Note that the interior of an angle is a convex set since it is the intersection of two convex sets.

Definition 6.9 *Three points A, B, and C are said to be* collinear *if there exists one line ℓ so that $A, B, C \in \ell$. The points are* noncollinear *otherwise.*

Please note that by our first axiom every pair of points is collinear. Therefore, we do not talk about collinearity except for any number of points greater than two.

Also, note that if A, B, and C are non-collinear, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are neither opposite nor equal.

Definition 6.10 *Let A, B, and C be three non-collinear points. The* triangle $\triangle ABC$ *consists of the union of the three segments AB, BC, and AC:*

$$\triangle ABC = AB \cup AC \cup BC.$$

The points A, B, and C are called the vertices *of the triangle and the segments are called the* sides.

Theorem 6.3 (Pasch's Theorem) *Let $\triangle ABC$ be a triangle and let ℓ be a line such that none of A, B, and C lies on ℓ. If ℓ intersects AB, then ℓ also intersects either AC or BC.*

PROOF: Let $\triangle ABC$ be given and let ℓ be a line satisfying the above condition. Let H_1 and H_2 be the two half-planes determined by ℓ. The points A and B are on opposite sides of ℓ. Without loss of generality, we may assume that $A \in H_1$ and $B \in H_2$. Now C is either in H_1 or in H_2. If $C \in H_1$ then $BC \cap \ell \neq \emptyset$ and if $C \in H_2$ then $AC \cap \ell \neq \emptyset$, and we are done.∎

6.1.5 Angle Measure

Definition 6.11 *A ray $\overrightarrow{AD}$ is* between *rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ if D is in the interior of $\angle BAC$.*

Axiom 5 (The Protractor Axiom) *For every angle $\angle BAC$ there is a number $m(\angle BAC)$, called the* measure of $\angle BAC$*, such that:*

i) $\angle 0^\circ \leq m(\angle BAC) \leq \angle 180^\circ$*, for every angle $\angle BAC$;*

ii) $m(\angle BAC) = 0$ *if and only if* $\overrightarrow{AB} = \overrightarrow{AC}$*;*

iii) For each real number r, $0 < r < 180$ and for each half-plane H bounded by $\overleftrightarrow{AB}$ there is a unique ray $\overrightarrow{AD}$ so that $D \in H$ and $m(\angle BAD) = \angle r^\circ$;

iv) If $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ then

$$m(\angle BAD) + m(\angle DAC) = m(\angle BAC).$$

Definition 6.12 *We say that angles $\angle BAC$ and $\angle EDF$ are* congruent, $\angle BAC \cong \angle EDF$*, if $m(\angle BAC) = m(\angle EDF)$.*

Definition 6.13 *Angle $\angle BAC$ is a* right angle *if $m(\angle BAC) = \angle 90^\circ$. $\angle BAC$ is an* acute angle *if $m(\angle BAC) < \angle 90^\circ$. $\angle BAC$ is an* obtuse angle *if $m(\angle BAC) > \angle 90^\circ$.*

6.2 Betweenness, the Crossbar Theorem, and Neutral Results

While we do not want to become bogged down in dealing with the issues in the foundations of geometry, we do need to insure that what we are doing does have a sound foundation. We are interested in seeing what the different geometries look like, but we need to know that what we have based this on is, in fact, reasonable and verifiable.

To this end, we will list a number of theorems and definitions, some with proof and some without. These will form the base for our investigations, though we will not go over every proof. The results are here when we need to refer to them later.

6.2.1 Betweenness

From the Ruler Axiom we can define a function from a geometric line to the real line that will be quite useful. A one-to-one correspondence $f\colon \ell \to \mathbb{R}$ such that $d(P,Q) = |f(P) - f(Q)|$ for every point $P, Q \in \ell$ is called a *coordinate function* for the line ℓ and the number $f(P)$ is called the *coordinate of the point* P.

Theorem 6.4 *Let ℓ be a line; let A, B, and C be three distinct points on ℓ; and let $f\colon \ell \to \mathbb{R}$ be a coordinate function for ℓ. The point C is between A and B if and only if either $f(A) < f(B) < f(C)$ or $f(A) > f(B) > f(C)$.*

Lemma 6.1 *If A, B, and C are three distinct collinear points, then exactly one of them lies between the other two.*

Definition 6.14 *Let A and B be two distinct points. The point M is a* midpoint *of AB if $A * M * B$ and $AM \cong MB$.*

Theorem 6.5 *If A and B are distinct points, then there exists a unique point M such that M is the midpoint of AB.*

Theorem 6.6 *Let ℓ be a line, let $A \in \ell$, and let B be an external point for ℓ. If C is a point between A and B, then B and C are on the same side of ℓ.*

Lemma 6.2 *Let ℓ be a line, let $A \in \ell$, and let B be an external point for ℓ. If $C \in \overrightarrow{AB}$ and $C \neq A$, then B and C are on the same side of ℓ.*

Lemma 6.3 *Betweenness for rays is well-defined. (If D is in the interior of $\angle BAC$ then every point on the ray $\overrightarrow{AD}$ is in the interior of $\angle BAC$.)*

Lemma 6.4 (The Z-Theorem) *Let ℓ be a line and let $A, D \in \ell$ be distinct points. If B and E are on opposite sides of ℓ then $\overrightarrow{AB} \cap \overrightarrow{DE} = \emptyset$.*

Theorem 6.7 *Let A, B, and C be three noncollinear points and let D be a point on the line $\overleftrightarrow{BC}$. The point D is between points B and C if and only if the ray $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.*

Theorem 6.8 (Betweenness Theorem for Rays) *Let A, B, C, and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$. Then $m(\angle BAD) < m(\angle BAC)$ if and only if $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.*

Theorem 6.9 (Angle Bisectors) *Let A, B, and C be three noncollinear points. There exists a unique ray $\overrightarrow{AD}$ in the interior of $\angle BAC$ such that $m(\angle BAD) = m(\angle DAC)$. The ray $\overrightarrow{AD}$ is called the* angle bisector.

This gives us the machinery we need to prove an important theorem that Euclid and many other early geometers took for granted. It seems obvious, but without some of the prior work we could not know that a ray emanating from a vertex would actually intersect a segment running from one side of an angle to the other. The segment is called a "crossbar".

Theorem 6.10 (Crossbar Theorem) *Given $\triangle ABC$, let D be a point in the interior of $\angle BAC$. There is a point G so that G lies on both $\overrightarrow{AD}$ and BC.*

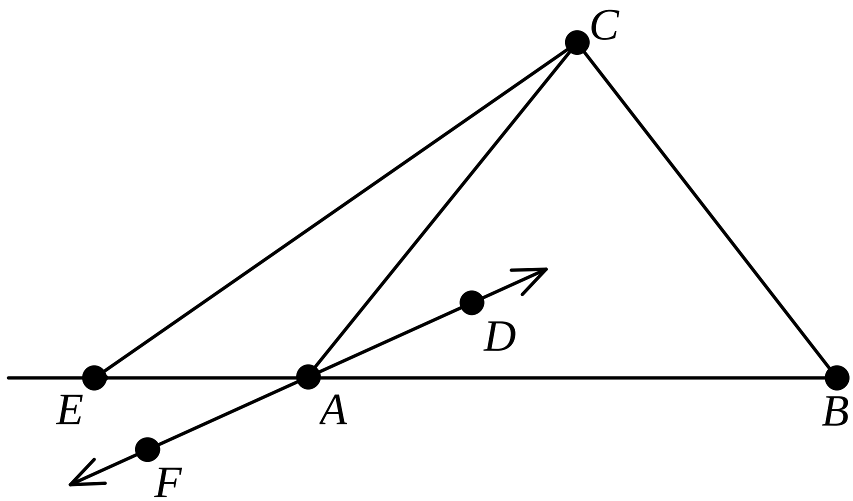


Figure 6.1: Crossbar Theorem

Proof: We are given a ray $\overrightarrow{AD}$ between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Let us use proof by contradiction and assume that $\overrightarrow{AD} \cap BC = \emptyset$.

Let $\overrightarrow{AF}$ be the opposite ray to $\overrightarrow{AD}$. If $\overrightarrow{AF} \cap BC = P$, then $B * P * C$ and by Theorem 6.6 we have that P lies in the interior of $\angle CAB$. However, this contradicts Lemma 6.2, which says that no point on the opposite ray can be interior to the angle. Thus, we have that $\overrightarrow{AF} \cap BC = \emptyset$. Now, this means that $\overleftrightarrow{AD} \cap BC = \emptyset$ by since neither $\overrightarrow{AD}$ nor its opposite ray intersect BC. It follows that B and C are on the same side of the line $\overleftrightarrow{AD}$.

Let E be a point on the line $\overleftrightarrow{AC}$ so that $C * A * E$. Then, C and E are on opposite sides of $\overleftrightarrow{AD}$, and by the Z Theorem, B and E are on opposite sides of $\overleftrightarrow{AD}$. Then we know that B is in the interior of $\angle DAE$, which means that B and E are on the same side of the line $\overleftrightarrow{AD}$. We now have a contradiction.

Thus, we have that $\overrightarrow{AD} \cap BC \neq \emptyset$. ∎

This can be summed up more succinctly in the following theorem.

Theorem 6.11 *A point D is in the interior of angle $\angle BAC$ if and only if the ray $\overrightarrow{AD}$ intersects the interior of segment BC.*

Definition 6.15 *Two angles $\angle BAD$ and $\angle DAC$ form a linear pair if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.*

Theorem 6.12 (Linear Pair Theorem) *If angles $\angle BAD$ and $\angle BAC$ forma a linear pair, then $m(\angle BAD) + m(\angle DAC) = \angle 180°$.*

Definition 6.16 *Two angles $\angle BAC$ and $\angle EDF$ are* supplementary *if $m(\angle BAC) + m(\angle EDF) = \angle 180°$.*

Definition 6.17 *Two lines ℓ and m are* perpendicular *if there is a point $A \in \ell \cap m$ and points $B \in \ell$ and $C \in m$ so that $\angle BAC$ is a right angle. This will be denoted by $\ell \perp m$.*

Definition 6.18 *If A and B are distinct points, a* perpendicular bisector *of AB is a line ℓ through the midpoint of AB so that $\overleftrightarrow{AB} \perp \ell$.*

6.2.2 How do Angle Measure and Distance Interact?

We have axioms that tell us about congruent angles and congruent segments. Are these at all connected? In order to see if there is a connection, we need to look at objects that contain both segments and angles. The simplest of these objects are triangles. We will look there for any relationships.

Definition 6.19 *Two triangles are said to be* congruent *if there is a one-to-one correspondence of the vertices of the two triangles such that the corresponding angles are congruent and the corresponding sides are congruent.*

You need to note that our definition says that the way to tell if two triangles are congruent is to find the six congruences that are in the definition: three angle congruences and three segment congruences.

We are used to having the Side-Angle-Side result to use in our geometries. Euclid proved it as his fourth proposition (though there are some problems with the proof). We would like to be able to do the same, but the five axioms that we have so far are not strong enough to imply the SAS proposition. There are geometries that satisfy our five axioms but not the SAS proposition. Thus, we must add it as another axiom to our list.

Axiom 6 (Side-Angle-Side Axiom or SAS) *If $\triangle ABC$ and $\triangle DEF$ are triangles so that $AB \cong DE$, $\angle ABC \cong \angle DEF$ and $BC \cong EF$, then $\triangle ABC \cong \triangle DEF$.*

6.3 Neutral Geometry

We have not spent too much time considering the ramifications of the axioms unrelated to the Parallel Axiom. What can we derive from these alone? Remember, the purpose of a lot of mathematics in the time between Euclid and Bolyai-Lobachevskii-Gauss was to prove that the Parallel Postulate did depend on the others.

6.3.1 Angle Side Angle Theorem

While we could use SAS only to show congruence of triangles, it is sometimes easier to work with other information.

Theorem 6.13 (Angle-Side-Angle, ASA) *Given triangles $\triangle ABC$ and $\triangle DEF$ with $\angle CAB \cong \angle FDE$, $AB \cong DE$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Proof: Let $\triangle ABC$ and $\triangle DEF$ be as given. There exists a point $C' \in \overrightarrow{AC}$ so that $AC' \cong DF$ by the Ruler Axiom. Then by SAS $\triangle ABC' \cong \triangle DEF$ and so $\angle ABC' \cong \angle DEF$ by CPCTC. We were given that $\angle ABC \cong \angle DEF$ so we must have that $\angle ABC \cong \angle ABC'$. Then by the Protractor Axiom we must have that $\overrightarrow{BC} = \overrightarrow{BC'}$. However, $\overrightarrow{BC}$ can only intersect $\overleftrightarrow{AC}$ in one point, so $C = C'$ and we are done. ∎

This leads us to the result

Theorem 6.14 *If in $\triangle ABC$ we have $\angle ABC \cong \angle ACB$ then $AB \cong AC$.*

This next result will turn out to be one of the most important in terms of how we will be using it throughout the course.

Theorem 6.15 (Existence of Perpendiculars) *For every line ℓ and for every external point P, there exists a line m through P so that $\ell \perp m$.*

From this point on we will use the terminology "drop a perpendicular from P to ℓ" meaning that we are invoking this result without referring to it by name and number.

PROOF: Let ℓ be a line and P a point not on ℓ. There are at least two distinct points $A, B \in \ell$. By the Protractor Axiom there is a point Q on the opposite side of ℓ from P so that $\angle PAB \cong \angle QAB$. There is a point $P' \in \overrightarrow{AQ}$ so that $AP' \cong AP$ by the Ruler Axiom. Let $m = \overleftrightarrow{PP'}$. Since P and P' are on opposite sides of ℓ, $PP' \cap \ell = \{F\}$.

If $A = F$, then $\angle BFP$ and $\angle BFP'$ supplementary and congruent, so they must be right angles and $\ell \perp m$.

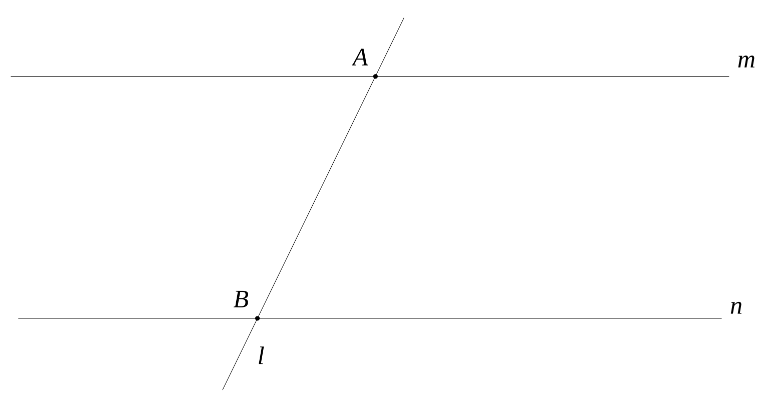
If $A \neq F$ then $\triangle FAP \cong \triangle FAP'$ by SAS, so $\angle AFP \cong \angle AFP'$ and they are supplements by construction, so they are right angles and $m \perp \ell$. ∎

6.3.2 Alternate Interior Angles

Definition 6.20 Let $\mathscr{L}$ be a set of lines in the plane. A line ℓ is **transversal** of $\mathscr{L}$ if

1. $\ell \notin \mathscr{L}$, and
2. $\ell \cap m \neq \emptyset$ for all $m \in \mathscr{L}$.

Let ℓ be transversal to m and n at points A and B, respectively. We say that each of the angles of intersection of ℓ and m and of ℓ and n has a *transversal side* in ℓ and a *non-transversal side* not contained in ℓ.



Definition 6.21 An angle of intersection of m and ℓ and one of n and ℓ are **alternate interior angles** if their transversal sides are opposite directed and intersecting, and if their non-transversal sides lie on opposite sides of ℓ. Two of these angles are **corresponding angles** if their transversal sides have like directions and their non-transversal sides lie on the same side of ℓ.

Theorem 6.16 (Alternate Interior Angle Theorem) *If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.*

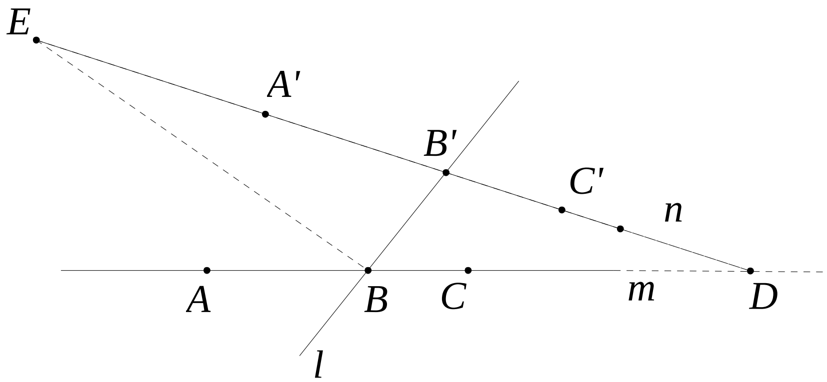


Figure 6.2: Alternate Interior Angles

PROOF: Let m and n be two lines cut by the transversal ℓ. Let the points of intersection be B and B', respectively. Choose a point A on m on one side of ℓ, and choose $A' \in n$ on the same side of ℓ as A. Likewise, choose $C \in m$ on the opposite side of ℓ from A. Choose $C' \in n$ on the same side of ℓ as C. Then it is on the opposite side of ℓ from A'.

We are given that $\angle A'B'B \cong \angle CBB'$. Assume that the lines m and n are not parallel; *i.e.*, they have a nonempty intersection. Let us denote this point of intersection by D. D is on one side of ℓ, so by changing the labeling, if necessary, we may assume that D lies on the same side of ℓ as C and C'. There is a unique point E on the ray $B'A'$ so that $B'E \cong BD$. Since, $BB' \cong BB'$, we may apply the SAS Axiom to prove that

$$\triangle EBB' \cong \triangle DBB'.$$

From the definition of congruent triangles, it follows that $\angle DB'B \cong \angle EBB'$. Now, the supplement of $\angle DBB'$ is congruent to the supplement of $\angle EB'B$. The supplement of $\angle EB'B$ is $\angle DB'B$ and $\angle DB'B \cong \angle EBB'$. Therefore, $\angle EBB'$ is congruent to the supplement of $\angle DBB'$. Since the angles share a side, they are themselves supplementary. Thus, $E \in n$ and we have shown that $\{D, E\} \subset n$ or that $m \cap n$ is more that one point. This contradiction gives us that m and n must be parallel. ∎

Corollary 1 *If m and n are distinct lines both perpendicular to the line ℓ, then m and n are parallel.*

PROOF: ℓ is the transversal to m and n. The alternate interior angles are right angles. All right angles are congruent, so the *Alternate Interior Angle Theorem* applies. m and n are parallel. ∎

Corollary 2 *If P is a point not on ℓ, then the perpendicular dropped from P to ℓ is unique.*

PROOF: Assume that m is a perpendicular to ℓ through P, intersecting ℓ at Q. If n is another perpendicular to ℓ through P intersecting ℓ at R, then m and n are two distinct lines perpendicular to ℓ. By the above corollary, they are parallel, but each contains P. Thus, the second line cannot be distinct, and the perpendicular is unique. ∎

The point at which this perpendicular intersects the line ℓ, is called the *foot* of the perpendicular.

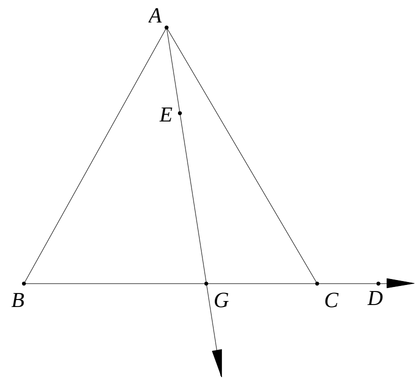


Figure 6.3: Exterior Angle Theorem

Corollary 3 *If ℓ is any line and P is any point not on ℓ, there exists at least one line m through P which does not intersect ℓ.*

PROOF: By Corollary 2 there is a unique line, m, through P perpendicular to ℓ. Now there is a unique line, n, through P perpendicular to m. By Corollary 1 ℓ and n are parallel. ∎

Note that while we have proved that there is a line through P which does not intersect ℓ, we have not (and cannot) proved that it is *unique*.

6.3.3 Weak Exterior Angle Theorem

Let $\triangle ABC$ be any triangle in the plane. This triangle gives us not just three segments, but in fact three lines.

Definition 6.22 An angle supplementary to an angle of a triangle is called an **exterior angle** of the triangle. The two angles of the triangle not adjacent to this exterior angle are called the **remote interior angles**.

Theorem 6.17 (Exterior Angle Theorem) *An exterior angle of a triangle is greater than either remote interior angle. (See Figure 6.3)*

PROOF: We shall show that $\angle ACD > \angle A$. In a like manner, you can show that $\angle ACD > \angle B$. Then by using the same techniques, you can prove the same for the other two exterior angles.

Now, either:

$$\angle A < \angle ACD \quad \angle A \cong \angle ACD \quad \text{or } \angle A > \angle ACD.$$

If $\angle A = \angle BAC \cong \angle ACD$, then by the *Alternate Interior Angle Theorem*, lines AB and CD are parallel. This is impossible, since they both contain B.

Assume, then, that $\angle A > \angle ACD$. Then there exists a ray $\overrightarrow{AE}$ between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ so that

$$\angle CAE \cong \angle ACD.$$

By the *Crossbar Theorem*, ray $\overrightarrow{AE}$ intersects BC in a point G. Again by the *Alternate Interior Angle Theorem* lines AE and CD are parallel. This is a contradiction.

Thus, $\angle A < \angle ACD$. ∎

Lemma 6.5 (AAS Congruence) *In triangles $\triangle ABC$ and $\triangle DEF$ given that $AC \cong DF$, $\angle A \cong \angle D$, and $\angle B \cong \angle E$, then $\triangle ABC \cong \triangle DEF$.*

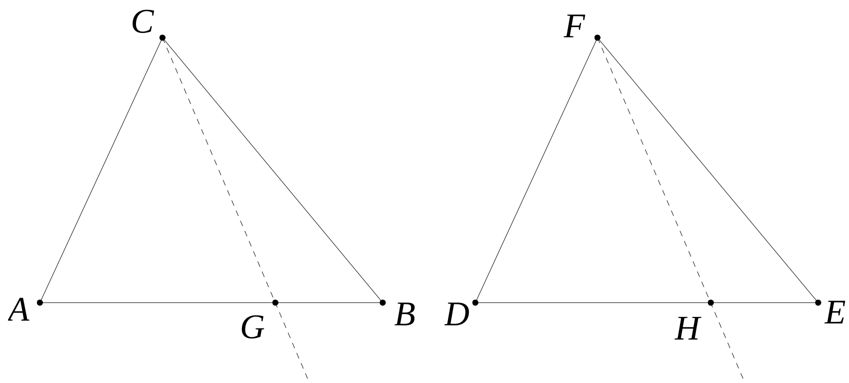


Figure 6.4: SAA Congruence

Proof: If $AB \cong DE$, we are done by *Angle-Side-Angle*. Thus, let us assume that $AB \ncong DE$. Then, by we must have that either $AB < DE$ or $AB > DE$.

If $AB < DE$, then there is a point $H \in DE$ so that $AB \cong DH$. Then by the *SAS Theorem* $\triangle ABC \cong \triangle DHF$. Thus, $\angle B \cong \angle DHF$. But $\angle DHF$ is exterior to $\triangle FHE$, so by the *Exterior Angle Theorem* $\angle DHF > \angle E \cong \angle B$. Thus, $\angle DHF > \angle B$, and we have a contradiction. Therefore, AB is not less than DE. By a similar argument, we can show that assuming that $AB > DE$ leads to a similar contradiction.

Thus, our hypothesis that $AB \ncong DE$ cannot be valid. Thus, $AB \cong DE$ and $\triangle ABC \cong \triangle DEF$ by ASA. ∎

Lemma 6.6 *Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.*

Lemma 6.7 (SSS Congruence) *In triangles $\triangle ABC$ and $\triangle DEF$ given that $AC \cong DF$, $AB \cong DE$, and $BC \cong EF$, then $\triangle ABC \cong \triangle DEF$.*

Lemma 6.8 *Every segment has a unique midpoint.*

Proof: Let AB be any segment in the plane, and let C be any point not on line AB. There exists a unique ray $\overrightarrow{BX}$ on the opposite side of line AB from P such that $\angle PAB \cong \angle XBA$. There is a unique point Q on the ray $\overrightarrow{BX}$ so that $AP \cong BQ$. Q is on the opposite side of line AB from P. Since P and Q are on opposite sides of line AB, $PQ \cap AB \neq \emptyset$. Let M denote this point of intersection. Either M lies between A and B, A lies between M and B, B lies between A and M, $M = A$, or $M = B$.

We want to show that M lies between A and B, so assume not. Since $\angle PAB \cong \angle QBA$, by construction, we have from the *Alternate Interior Angle Theorem* that lines AP and BQ are parallel. If $M = A$ then A, P, and M are collinear on the line AP and lines $AP = AB$ which intersects line BQ. We can dispose of the case $M = B$ similarly.

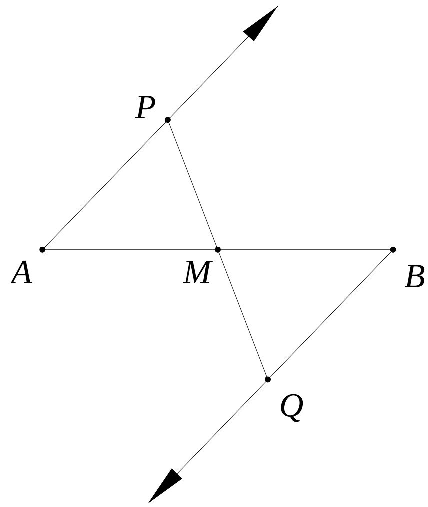


Figure 6.5: Uniqueness of the midpoint

Thus, assume that A lies between M and B. This will mean that the line PA will intersect side MB of $\triangle MBQ$ at a point between M and B. Thus, by *Pasch's Theorem* it must intersect either MQ or BQ. It cannot intersect side BQ as lines AP and BQ are parallel. If line AP intersects MQ then it must contain MQ for P, Q, and M are collinear. Thus, $M = A$ which we have already shown is impossible. Thus, we have shown that A cannot lie between M and B.

In the same manner, we can show that B cannot lie between A and M. Thus, we have that M must lie between A and B. This means that $\angle AMP \cong \angle BMQ$ since they are vertical angles. By *Angle-Angle-Side* we have that $\triangle AMP \cong \triangle BMQ$. Thus, $AM \cong MB$ and M is the midpoint of AB. ∎

We state the following results without proof. The proof is left to the reader.

Lemma 6.9 *In a triangle $\triangle ABC$ the greater angle lies opposite the greater side and the greater side lies opposite the greater angle;* i.e., *$AB > BC$ if and only if $\angle C > \angle A$.*

Lemma 6.10 *Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\angle B < \angle B'$ if and only if $AC < A'C'$.*

6.4 Saccheri-Legendre Theorem

Corollary 1 *The sum of the degree measures of any* two *angles of a triangle is less than $180°$.*

This follows from the Exterior Angle Theorem.

PROOF: We want to show that $\angle A + \angle B < 180°$. From the Exterior Angle Theorem,

$$\angle A < \angle CBD$$
$$\angle A + \angle B < \angle CBD + \angle B = 180°,$$

since they are supplementary angles. ∎

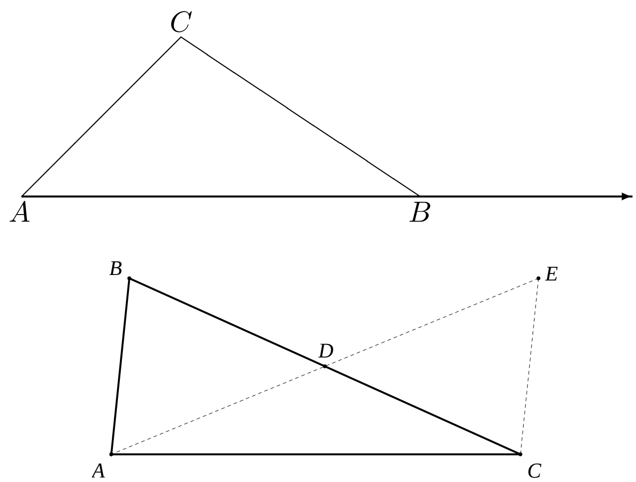


Figure 6.6: First step in Saccheri-Legendre Theorem

Corollary 2 (Triangle Inequality) *If A, B, and C are three noncollinear points, then* $|AC| < |AB| + |BC|$.

Theorem 6.18 (Saccheri-Legendre Theorem) *The sum of the degree measures of the three angles in any triangle is* less than or equal to $180°$*;*

$$\angle A + \angle B + \angle C \leq 180°.$$

PROOF: Let us assume not; *i.e.*, assume that we have a triangle $\triangle ABC$ in which $\angle A + \angle B + \angle C > 180°$. So there is an $x \in \mathbb{R}^+$ so that

$$\angle A + \angle B + \angle C = 180° + x.$$

Compare Figure 6.6. Let D be the midpoint of BC and let E be the unique point on the ray AD so that $DE \cong AD$. Then by SAS $\triangle BAD \cong \triangle CED$. This makes

$$\angle B = \angle DCE \qquad \angle E = \angle BAD.$$

Thus,

$$\begin{aligned} \angle A + \angle B + \angle C &= (\angle BAD + \angle EAC) + \angle B + \angle ACB \\ &= \angle E + \angle EAC + (\angle DCE + \angle ACD) \\ &= \angle E + \angle A + \angle C \end{aligned}$$

So, $\triangle ABC$ and $\triangle ACE$ have the same angle sum, even though they need not be congruent. Note that $\angle BAE + \angle CAE = \angle BAC$, hence

$$\angle CEA + \angle CAE = \angle BAC.$$

It is impossible for both of the angles $\angle CEA$ and $\angle CAE$ to have angle measure greater than $1/2\angle BAC$, so at least one of the angles has angle measure less than or equal to $1/2\angle BAC$.

Therefore, there is a triangle $\triangle ACE$ so that the angle sum is $180° + x$ but in which one angle has measure less than or equal to $1/2\angle A°$. Repeat this construction to get another triangle with angle sum $180° + x$ but in which one angle has measure less than or equal to $1/4\angle A°$. Now there is an $n \in \mathbb{Z}^+$ so that

$$\frac{1}{2^n}\angle A \leq x,$$

by the Archimedean property of the real numbers. Thus, after a finite number of iterations of the above construction we obtain a triangle with angle sum $180° + x$ in which one angle has measure less than or equal to

$$\frac{1}{2^n}\angle A \leq x.$$

Then the other two angles must sum to a number greater than 180° contradicting Corollary 1. ∎

Corollary 1 *In $\triangle ABC$ the sum of the degree measures of two angles is less than or equal to the degree measure of their remote exterior angle.*

6.4.1 The Defect of a Triangle

Since the angle sum of any triangle in neutral geometry is not more than 180°, we can compute the difference between the number 180 and the angle sum of a given triangle.

Definition 6.23 The **defect** of a triangle $\triangle ABC$ is the number

$$\delta(ABC) = \textit{defect}(\triangle ABC) = 180° - (\angle A + \angle B + \angle C).$$

In euclidean geometry we are accustomed to having triangles whose defect is zero. Is this always the case? The Saccheri-Legrendre Theorem indicates that it may not be so. Is the *defect* of triangles preserved? That is, if we have one defective triangle, then are all of the sub and super-triangles defective? By defective, we mean that the triangles have positive defect.

Theorem 6.19 (Additivity of Defect) *Let $\triangle ABC$ be any triangle and let D be a point between A and B. Then $\delta(ABC) = \delta(ACD) + \delta(BCD)$.*

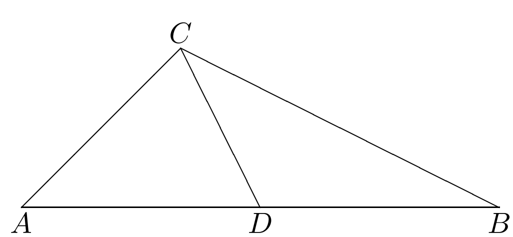


Figure 6.7: Additivity of Defect

Proof: Since the ray CD lies in $\angle ACB$, we know that

$$\angle ACB = \angle ACD + \angle BCD,$$

and since $\angle ADC$ and $\angle BDC$ are supplementary angles $\angle ADC + \angle BDC = 180^\circ$. Therefore,

$$\begin{aligned}\delta(ABC) &= 180^\circ - (\angle A + \angle B + \angle C)\\ &= 180^\circ - (\angle A + \angle B + \angle ACD + \angle BCD)\\ &= 180^\circ + 180^\circ - (\angle A + \angle B\\ &\qquad + \angle ACD + \angle BCD + \angle ADC + \angle BDC)\\ &= \delta(ACD) + \delta(BCD).\end{aligned}$$

∎

Corollary 1 *$\delta(ABC) = 0$ if and only if $\delta(ACD) = \delta(BCD) = 0$.*

A **rectangle** is a quadrilateral all of whose angles are right angles. We cannot prove the existence or non-existence of rectangles in Neutral Geometry. Nonetheless, the following result is extremely useful.

Theorem 6.20 *If there exists a triangle of defect 0, then a rectangle exists. If a rectangle exists, then every triangle has defect 0.*

Let us first outline the proof in five steps.

1. Construct a *right* triangle having defect 0.
2. From a right triangle of defect 0, construct a rectangle.
3. From one rectangle, construct arbitrarily large rectangles.
4. Prove that all *right* triangles have defect 0.
5. If every *right* triangle has defect 0, then *every* triangle has defect 0.

Having outlined the proof, each of the steps is relatively straightforward.

1. Construct a *right* triangle having defect 0.
   Let us assume that we have a triangle $\triangle ABC$ so that $\delta(ABC) = 0$. We may assume that $\triangle ABC$ is not a right triangle, or we are done. Now, at least two angles are acute since the angle sum of any two angles is always less than 180°. Let us assume that $\angle A$ and $\angle B$ are acute. Also, let D be the foot of C on line AB. We need to know that D lies between A and B.

   Assume not; *i.e.*, assume that A lies between D and B. (See Figure **??**.) This means that $\angle CAB$ is exterior to $\triangle CAD$ and, therefore, $\angle A > \angle CDA = 90^\circ$. This makes $\angle A$ obtuse, a contradiction. Similarly, if B lies between A and D we can show that $\angle B$ is obtuse. Thus, we must have that D lies between A and B.

   This makes $\triangle ADC$ and $\triangle BDC$ right triangles. By Corollary 1 above, since $\triangle ABC$ has defect 0, each of them has defect 0, and we have two right triangles with defect 0.

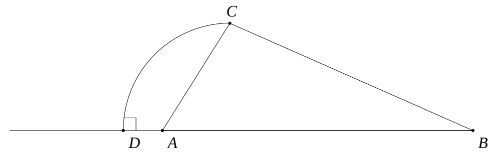


Figure 6.8: Right triangle defect

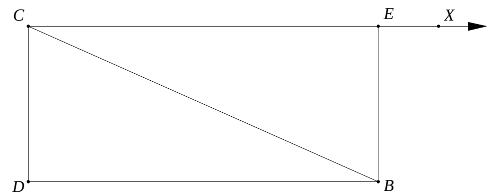


Figure 6.9: Existence of Rectangles

2. From a right triangle of defect 0, construct a rectangle.
   We now have a right triangle of defect 0. Take $\triangle CBD$ from Step 1, which has a right angle at D. There is a unique ray CX on the opposite side of BC from D so that

$$\angle DBC \cong \angle BCX.$$

   Then there is a unique point E on ray CX such that $CE \cong BD$.

   Thus, $\triangle CDB \cong \triangle BEC$ by *SAS*. Then $\angle BEC = 90°$ and $\triangle BEC$ must also have defect 0. Now, clearly, since $\mathit{defect}(\triangle CDB) = 0$

$$\angle DBC + \angle BCD = 90°$$

   and, hence,

$$\angle ECB + \angle BCD = \angle ECD = 90°.$$

   Likewise, $\angle EBD = 90°$ and $\square CDBE$ is a rectangle.

3. From one rectangle, construct arbitrarily large rectangles.
   Given any right triangle $\triangle XYZ$, we can construct a rectangle $\square PQRS$ so that $PS > XZ$ and $RS > YZ$. By applying Archimedes Axiom, we can find a number n so that we copy segment BD in the above rectangle on the ray ZX to reach the point P so that $n \cdot BD \cong PZ$ and X lies between P and Z. We make n copies of our rectangle sitting on $PZ = PS$. This gives us a rectangle with vertices P, $Z = S$, Y, and some other point. Now, using the same technique, we can find a number m and a point R on the ray ZY so that $m \cdot BE \cong RZ$ and Y lying between R and Z. Now, constructing m copies of the long rectangle, gives us the requisite rectangle containing $\triangle XYZ$.

4. Prove that all *right* triangles have defect 0.
   Let $\triangle XYZ$ be an arbitrary right triangle. By Step 3 we can embed it in a rectangle $\square PQRS$.

   Since $\triangle PQR \cong \triangle PSR$, we have that $\angle RPS + \angle PRS = 90°$ and then, $\triangle PRS$ has defect 0. Using Corollary 1 to Theorem 6.19 we find $\mathit{defect}(\triangle RXY) = 0$ and thus, $\mathit{defect}(\triangle XYZ) = 0$. Therefore, each triangle has defect 0.

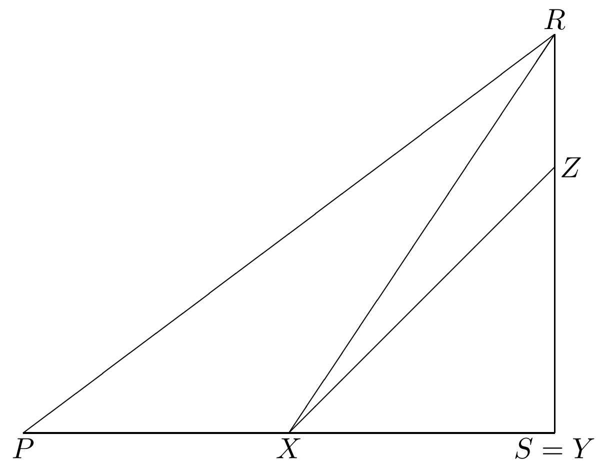


Figure 6.10: Step 1

5. If every *right* triangle has defect 0, then *every* triangle has defect 0.
As in the first step, use the foot of a vertex to decompose the triangle into two right triangles, each of which has defect 0, from Step 4. Thus, the original triangle has defect 0.

Corollary 1 *If there is a triangle with positive defect, then all triangles have positive defect.*