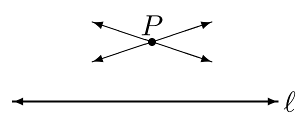
Chapter 7

We Choose Many Parallels!

7.1 Hyperbolic Axiom Results

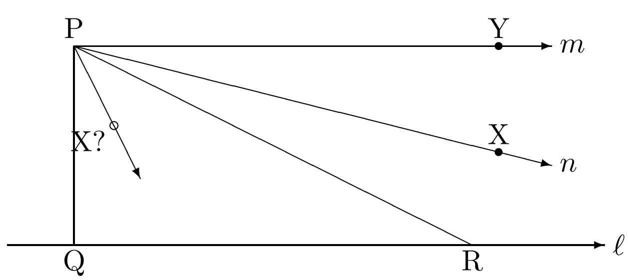
Hyperbolic geometry is often called *Bolyai-Lobachevskiian geometry* after two of its *discovers* János Bolyai and Nikolai Ivanovich Lobachevskii. Bolyai first announced his discoveries in a 26 page appendix to a book by his father, the *Tentamen*, in 1831. Another of the great mathematicians who seems to have preceded Bolyai in his work is Karl Friedreich Gauss. He seems to have done some work in the area dating from 1792, but never published it. The first to publish a complete account of non-Euclidean geometry was Lobachevskii in 1829. It was first published in Russian and was not widely read. In 1840 he published a treatise in German.

We shall call our added axiom the Hyperbolic Axiom.



We shall denote the set of all points in the plane by $\mathbb{H}^2$, and call this the *hyperbolic plane.*

Lemma 7.1 *There exists a triangle whose angle sum is less than* $180°$*.*



PROOF: Let ℓ be a line and P a point not on ℓ such that two parallels to ℓ pass through P. We can construct one of these parallels as previously done using perpendiculars. Let Q be the foot of the perpendicular to ℓ through P. Let m be the perpendicular to the line PQ

through P. Then m and ℓ are parallel. Let n be another line through P which does not intersect ℓ. This line exists by the *Hyperbolic Axiom*. Let PX be a ray of n lying between PQ and a ray PY of m.

CLAIM: There is a point $R \in \ell$ on the same side of the line PQ as X and Y so that $\angle QRP < \angle XPY$.

PROOF OF CLAIM. The idea is to construct a sequence of angles

$$\angle QR_1P,\ \angle QR_2P,\ \ldots,\ \angle QR_nP,\ \ldots$$

so that $\angle QR_{j+1}P < \frac{1}{2}\angle QR_jP$. We will then apply Archimedes Axiom for real numbers to complete the proof.

There is a point $R_1 \in \ell$ so that $QR_1 \cong PQ$. Then $\triangle QR_1P$ is isosceles and $\angle QR_1P^\circ \leq 45^\circ$. Also, there is a point $R_2 \in \ell$ so that R_1 lies between F and R_2 and $R_1R_2 \cong PR_1$. Then $\triangle PR_1R_2$ is isosceles and $\angle R_1PR_2 \cong \angle QR_2P$. Since $\angle QR_1P$ is exterior to $\triangle PR_1R_2$ it follows that

$$\angle R_1PR_2 + \angle QR_2P \leq \angle QR_1P,$$

so then $\angle QR_2P \leq 22\frac{1}{2}^\circ$. Continuing with this construction, we find a point $R_n \in \ell$ so that R_{n-1} lies between A and R_n and

$$\angle QR_nP \leq \left(\frac{45}{2^n}\right)^\circ.$$

Applying the Archimedean axiom we see that for any positive real number, for example $\angle XPY$, there is a point $R \in \ell$ so that R is on the same side of the line PQ as X and Y and $\angle QRP < \angle XPY$. Thus, we have proved our claim.

Now, the ray PR lies in the interior of $\angle QPX$, for if not then the ray PX is in the interior of $\angle QRP$. By the *Crossbar Theorem* it follows that the ray $PX \cap \ell \neq \emptyset$ which implies that n and ℓ are not parallel—a contradiction. Thus, $\angle RPQ < \angle XPQ$. Then,

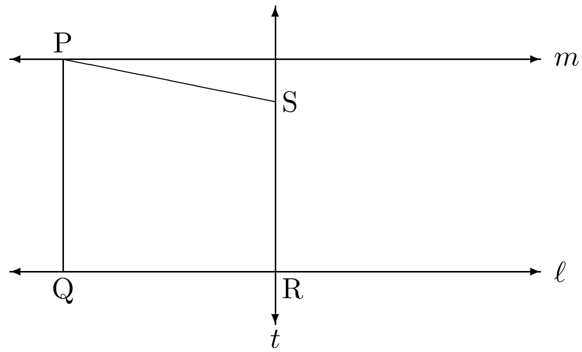
$$\angle RPQ + \angle QRP < \angle XPQ + \angle QRP < \angle XPQ + \angle XPY = 90^\circ.$$

Therefore, $\angle P + \angle Q + \angle R < 180^\circ$ and *defect*$(\triangle PQR) > 0$. ∎

The *Hyperbolic Axiom* only hypothesizes the existence of one line and one point not on that line for which there are two parallel lines. With the above theorem we can now prove a much stronger theorem.

Theorem 7.1 (Universal Hyperbolic Theorem) *In $\mathbb{H}^2$ for every line ℓ and for every point P not on ℓ there pass through P at least two distinct lines, neither of which intersect ℓ.*

PROOF: Drop a perpendicular PQ to ℓ and construct a line m through P perpendicular to PQ. Let R be any other point on ℓ, and construct a perpendicular t to ℓ through R. Now, let S be the foot of the perpendicular to t through P. Now, the line PS does not intersect ℓ since both are perpendicular to t. At the same time $PS \neq m$. Assume that $S \in m$, then $\square PQRS$ is a rectangle. By Theorem 6.20, if one rectangle exists all triangles have defect 0. We have a contradiction to Lemma 7.1. Thus, $PS \neq m$, and we are done. ∎



7.2 Angle Sums (again)

We have just proven the following theorem.

Theorem 7.2 *In $\mathbb{H}^2$ rectangles do not exist and all triangles have angle sum less than* $180°$.

This tells us that in hyperbolic geometry the defect of any triangle is a *positive* real number. We shall see that it is a very important quantity in hyperbolic geometry.

Corollary 1 *In $\mathbb{H}^2$ all convex quadrilaterals have angle sum less than* $360°$.

7.3 Saccheri Quadrilaterals

Girolamo Saccheri was a Jesuit priest who lived from 1667 to 1733. Before he died he published a book entitled *Euclides ab omni nævo vindicatus* (*Euclid Freed of Every Flaw*). It sat unnoticed for over a century and a half until rediscovered by the Italian mathematician Beltrami.

Saccheri wished to prove Euclid's Fifth Postulate from the other axioms. To do so he decided to use a *reductio ad absurdum* argument. He assumed the negation of the Parallel Postulate and tried to arrive at a contradiction. He studied a family of quadrilaterals that have come to be called *Saccheri quadrilaterals*. Let S be a convex quadrilateral in which two adjacent angles are right angles. The segment joining these two vertices is called the **base**. The side opposite the base is the **summit** and the other two sides are called the **sides**. If the sides are congruent to one another then this is called a **Saccheri quadrilateral**. The angles containing the summit are called the **summit angles**.

Theorem 7.3 *In a Saccheri quadrilateral*

i) the summit angles are congruent,

ii) the line joining the midpoints of the base and the summit—called the **altitude**—*is perpendicular to both.*

iii) the diagonals AC and BD are congruent, and

iv) $\square ABCD$ is a parallelogram.

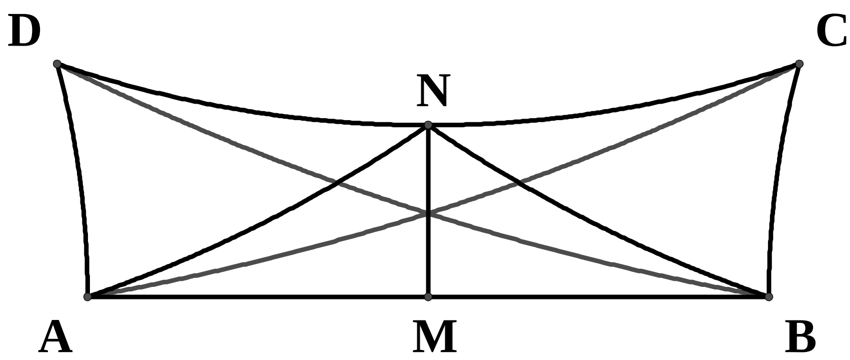


Figure 7.1: Saccheri Quadrilateral

Proof: Let M be the midpoint of AB and let N be the midpoint of CD.

1. We are given that
$$\angle DAB = \angle ABC = 90^\circ.$$
Now, $AD \cong BC$ and $AB \cong AB$, so that by SAS $\triangle DAB \cong \triangle CBA$, which implies that $BD \cong AC$. Also, since $CD \cong CD$ then we may apply the SSS criterion to see that $\triangle CDB \cong \triangle DCA$. Then, it is clear that $\angle D \cong \angle C$.

2. We need to show that the line MN is perpendicular to both lines AB and CD. Now $DN \cong CN$, $AD \cong BC$, and $\angle D \cong \angle C$. Thus by SAS $\triangle ADN \cong \triangle BCN$. This means then that $AN \cong BN$. Also, $AM \cong BM$ and $MN \cong MN$. By SSS $\triangle ANM \cong \triangle BNM$ and it follows that $\angle AMN \cong \angle BMN$. They are supplementary angles, hence they must be right angles. Therefore MN is perpendicular to AB. Using the analogous proof and triangles $\triangle DMN$ and $\triangle CMN$, we can show that MN is perpendicular to CD.

3. We proved that $AC \cong BD$ in the first part.

4. Since AB and CD have a common perpendicular, they are parallel. Since AD and BC have a common perpendicular (the base) they are parallel, so $\square ABCD$ is a parallelogram.

Thus, we are done. ∎

Lemma 7.2 *In a Saccheri quadrilateral the summit angles are acute.*

Proof: Recall from Corollary 1 to Theorem 7.2 that the angle sum for any convex quadrilateral is less that 360°. Thus, since the Saccheri quadrilateral is convex,

$$\begin{aligned} \angle A + \angle B + \angle C + \angle D &< 360^\circ \\ 2\angle C &< 180^\circ \\ \angle C &< 90^\circ \end{aligned}$$

Thus, $\angle C$ and $\angle D$ are acute. ∎

A convex quadrilateral three of whose angles are right angles is called a **Lambert quadrilateral**, cf. Figure 7.2.

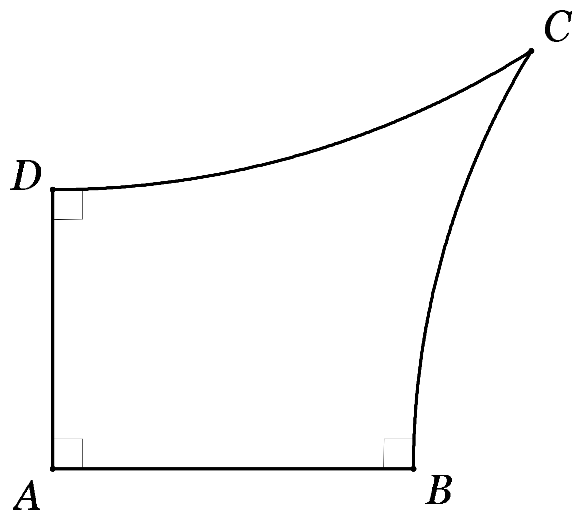


Figure 7.2: Lambert Quadrilateral

Lemma 7.3 *The fourth angle of a Lambert quadrilateral is acute.*

Proof: If the fourth angle were obtuse, our quadrilateral would have an angle sum greater than 360°, which cannot happen. If the angle were a right angle, then a rectangle would exist and all triangles would have to have defect 0. Since there is a triangle with angle sum less than 180°, we have a triangle with positive defect. Thus, the fourth angle cannot be a right angle either. ∎

Lemma 7.4 *The side adjacent to the acute angle of a Lambert quadrilateral is greater than its opposite side.*

Proof: We only need to show that $BC \neq AD$, since we already know that $BC \leq AD$. Assume that $BC \cong AD$. Then $\square ABCD$ is a Saccheri quadrilateral. Then, by Lemma 7.2 we must have that $\angle C \cong \angle D$, making $\square ABCD$ a rectangle. This contradicts Theorem 7.2, so $BC \neq AD$, making $BC < AD$. ∎

Lemma 7.5 *In a Saccheri quadrilateral the summit is greater than the base and the sides are greater than the altitude.*

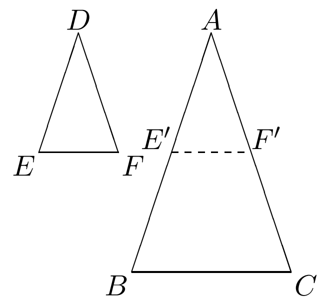
Proof: Using Theorem 7.3 if M is the midpoint of AB and N is the midpoint of CD, then $\square AMND$ is a Lambert quadrilateral. Thus, $AB > MN$ and, since $BC \cong AB$, both sides are greater than the altitude.

Also, applying Theorem 7.3 $DN > AM$. Since $CD \cong 2DN$ and $AB \cong 2AM$ it follows that $CD > AB$, so that the summit is greater than the base. ∎

7.4 Similar Triangles

In Euclidean geometry we are used to having two triangles similar if their angles are congruent. It is obvious that we can construct two non-congruent, yet similar, triangles. In fact John Wallis attempted to prove the Parallel Postulate of Euclid by adding another postulate.

WALLIS' POSTULATE: *Given any triangle $\triangle ABC$ and given any segment DE. There exists a triangle $\triangle DEF$ having DE as one of its sides that is similar to $\triangle ABC$.*



However Wallis' Postulate is equivalent to Euclid's Parallel Postulate. Thus, we know that the negation of Wallis' Postulate must hold in hyperbolic geometry. That is, under certain circumstances similar triangles do not exist. We can prove a much stronger statement.

Theorem 7.4 (AAA Criterion) *In $\mathbb{H}^2$ if $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$. That is, if two triangles are similar, then they are congruent.*

PROOF: Since $\angle BAC \cong \angle EDF$, there exists an isometry which sends D to A, the ray DE to the ray AB, and the ray DF to the ray AC. Let the image of E and F under this isometry be E' and F', respectively. If the two triangles are not congruent, then we may assume that $E' \neq B$ and that E' lies between A and B. Then BC and $E'F'$ cannot intersect by the Alternate Interior Angles Theorem. Then $BCE'F'$ forms a quadrilateral. The quadrilateral has the following angles:

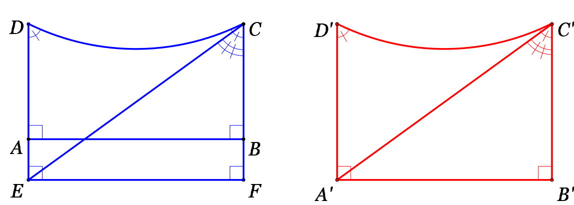
$$\begin{aligned} \angle E'BC &= \angle ABC \\ \angle F'CB &= \angle ACB \\ \angle BE'F' &= 180° - \angle ABC \\ \angle CF'E' &= 180° - \angle ACB \end{aligned}$$

which sum to 360°. This contradiction leads us to the fact that $E' = B$ and $F' = C$ and the two triangles are congruent. ∎

As a consequence of Theorem 7.4 we shall see that in hyperbolic geometry a segment can be determined with the aid of an angle. For example, an angle of an equilateral triangle determines the length of a side uniquely. Thus in hyperbolic geometry there is an *absolute unit of length.*

Theorem 7.5 *In $\mathbb{H}^2$ if $\square ABCD$ and $\square A'B'C'D'$ are two Saccheri quadrilaterals such that $\delta(\square ABCD) = \delta(\square A'B'C'D')$ and $CD \cong C'D'$, then $\square ABCD \cong \square A'B'C'D'$.*

PROOF: Given Saccheri quadrilaterals $\square ABCD$ and $\square A'B'C'D'$ as above. Since $\delta(\square ABCD) = \delta(\square A'B'C'D')$ we know that they have the same angle sum. Since the base angles of a Saccheri quadrilateral are right angles and the summit angles are congruent, we must have that $\angle ADC \cong \angle A'D'C' \cong \angle DCB \cong \angle D'C'B'$.



Choose a point E on $\overrightarrow{DA}$ and a point F on $\overrightarrow{CB}$ so that $DE \cong D'A' \cong CF$. We need to show that $\square EFCD \cong \square A'B'C'D'$ and then show that $A = E$ and $B = F$. This will complete the proof.

By SAS $\triangle EDC \cong \triangle A'D'C'$. Then by subtraction we have that $\angle ECF \cong \angle A'C'B'$. Therefore, by SAS $\triangle ECF \cong \triangle A'C'B'$. Therefore, $\angle EFC$ is a right angle. Similarly, we can show that $\angle FED$ is a right angle. Therefore, $\square EFCD \cong A'B'C'D'$.

Now, assume that $A \neq E$. Then we will have that $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$ since they are both perpendicular to $\overleftrightarrow{AD}$. Thus, $B \neq F$ and $\square ABFE$ is a rectangle. This contradicts the result that no rectangles exist, therefore $A = E$ and we are done. ∎

7.5 Common Perpendiculars

One of the common notions that we carry with us in our Euclidean outlook is about parallel lines. We tend to think of parallel lines as being lines that are everywhere equidistant, or *railroad tracks*. Is this notion of parallel strictly Euclidean, or can we transfer this concept to hyperbolic geometry?

Theorem 7.6 *If ℓ is a line, P a point not on ℓ, and m a line containing P. There exists at most one point $Q \in m$, $Q \neq P$ so that $d(P, \ell) = d(Q, \ell)$.*

The distance $d(P, \ell)$ is the length of the segment from P to the foot of the perpendicular in ℓ.

Proof: We are given ℓ and m and a point $P \in m$. Suppose that there are three distinct points $P, Q, R \in m$ so that $d(P, \ell) = d(Q, \ell) = d(R, \ell)$. Let P', Q', and R' denote the feet of P, Q, and R, respectively, in ℓ.

Now, none of these points lies on ℓ, since $d(P, \ell) > 0$. Therefore at least two of these points lie on the same side of ℓ. Assume that P and Q are on the same side of ℓ. Then $\square PP'Q'Q$ is a Saccheri quadrilateral and $\ell \parallel m$. Thus, all three of P, Q, and R are on the same side of ℓ.

Now, we know that one of P, Q, and R lies between the other two, so we may assume that $P * Q * R$. Then $\square PP'Q'Q$, $\square QQ'R'R$ and $\square PP'R'R$ are all Saccheri quadrilaterals. Therefore

$$\angle PQQ' \cong \angle QPP' \cong \angle PRR' \cong \angle RQQ'.$$

However, they are also supplements so

$$m(\angle PQQ') = m(\angle QPP') = 90°$$

and we have a rectangle. Since this is impossible, there could not be as many as three points that are equidistant from ℓ, and we are done. ∎

Definition 7.1 *Lines ℓ and m* admit a common perpendicular *if there exists a line n such that $n \perp m$ and $n \perp \ell$. If ℓ and m admit a common perpendicular n, then $n \cap \ell = \{P\}$ and $n \cap m = \{Q\}$ and the segment PQ is called the* common perpendicular segment *to ℓ and m.*

Theorem 7.7 *If ℓ and m are parallel lines and there are two points on m that are equidistant from ℓ, then m and ℓ admit a common perpendicular.*

Proof: Think Saccheri quadrilateral and that will give you the common perpendicular you seek. ∎

Theorem 7.8 *If ℓ and m admit a common perpendicular, then that common perpendicular is unique.*

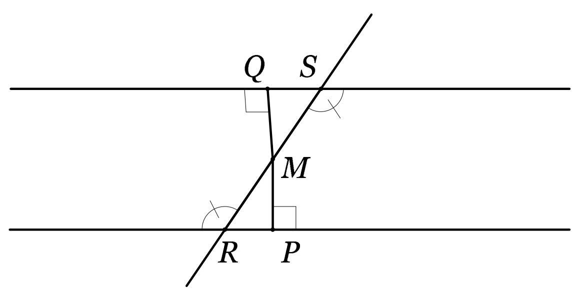
Proof: Again, think what two common perpendiculars would give you to prove this. ∎

Theorem 7.9 *Let ℓ and m be parallel lines cut by a transversal t. Alternate interior angles formed by ℓ and m with t are congruent if and only if ℓ and m admit a common perpendicular and t passes through the midpoint of the common perpendicular segment.*

This theorem tells us that the parallel lines and the transversal have to be extremely special in order for alternate interior angles to be congruent.

Proof: Let ℓ and m be parallel lines cut by the transversal t and let $\{R\} = t \cap \ell$ and $\{Q\} = t \cap m$.

First, assume that both pairs of alternate interior angles formed by ℓ and m with t are congruent. We will have to construct a common perpendicular to ℓ and m and show that t passes through the midpoint of the common perpendicular segment. If the alternate interior angles are right angles, then t is the common perpendicular and we are done. Thus, we may assume that these alternate interior angles are not right angles.



Let M be the midpoint of RS and drop a perpendicular from M to each of ℓ and m. Call these points P and Q. We do not know, *a priori*, that the point P, Q, and M are collinear. We must show this. Now, $\triangle RPM \cong \triangle SQM$ by AAS, so $\angle RMP \cong \angle SMQ$ and therefore $\overrightarrow{MP}$ and $\overrightarrow{MQ}$ are opposite rays and segment PQ is the common perpendicular segment for ℓ and m.

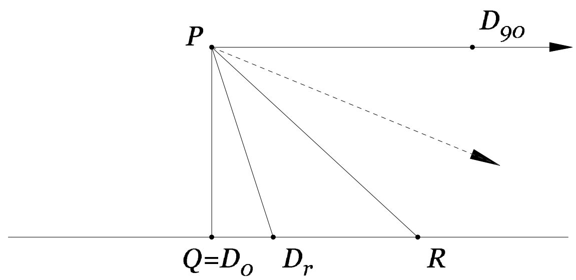
The proof of the other direction is left as an exercise. ∎

7.6 Asymptotically Parallel Lines

The previous section tells us that some parallel lines in hyperbolic geometry admit a single, unique, common perpendicular. Is it true that every pair of parallel lines in $\mathbb{H}^2$ admit a common perpendicular? It would seem as if this would be a nice characterization for parallel lines. Not everything is nice — at first.

Let ℓ be a line and $P \notin \ell$. Drop a perpendicular from P to ℓ and call the foot of P in ℓ the point Q. Let $R \in \ell$ so that $R \neq Q$. For each $r \in \mathbb{R}$ with $0 < r \leq 90$ there exists a point D_r on the same side of $\overleftrightarrow{PQ}$ as R, such that $m(\angle RPD_r) = r^\circ$ from the Protractor Axiom. Let

$$K = \{r \mid \overrightarrow{PD-r} \cap \overrightarrow{QR} \neq \emptyset\}.$$



Note that $K \subset \mathbb{R}$ and $K \neq \emptyset$ since $m(\angle RPQ) \in K$. Note also that K is bounded above by 90. By the Least Upper Bound Axiom, K has a least upper bound, $r_0 \leq 90$. It is also true that $90 \notin K$ since $\overrightarrow{PD_{90}}$ is parallel to ℓ.

We call K the *intersecting set* for P and $\overrightarrow{QR}$. We also call the number r_0 the *critical number* for P and $\overrightarrow{AB}$. Visually, think of the rays emanating from P starting with $\overrightarrow{PQ}$ and fanning out toward the ray $\overrightarrow{PD_{90}}$ that is parallel to ℓ. As the measure of $\angle QPD$ increases there will be a first ray that does not intersect $\overrightarrow{QR}$ and r_0 is the measure of the angle that that ray makes.

Theorem 7.10 *If $0 < r < r_0$ then $r \in K$. If $r_0 \leq r \leq 90$ then $r \notin K$.*

PROOF: Let r be given with $0 < r \leq 90$. Let us first assume that $r < r_0$. Now, r_0 is the least upper bound of K and $r < r_0$ so there must be a number $s \in K$ so that $r < s$, else r would be an upper bound that is smaller than the least upper bound. Since $s \in K$, the ray $\overrightarrow{PD_s}$ must intersect $\overrightarrow{AB}$. Call this point $T = \overrightarrow{AB} \cap \overrightarrow{PD_x}$. Since $r > s$, D_r lies in the interior of $\angle APD_s$. The Crossbar Theorem then tells us that PD_r must intersect AT, so $r \in K$.

Now, assume that $r \geq r_0$. Suppose that $r \in K$. Then $\overrightarrow{PD_r} \cap \overrightarrow{AB} = \{U\}$. Choose a point $T \in \overleftrightarrow{AB}$ so that $A * U * T$ and let $s = m(\angle APT)$. Then, by construction, $\overrightarrow{PD_s} \cap \overrightarrow{AB} \neq \emptyset$ and $s \in K$. Since $s > r$ from the Protractor Axiom, we have that $s > r_0$. This contradicts the fact that r_0 is an upper bound for K. This contradiction implies that $r \notin K$ and we are done. ∎

Definition 7.2 *Suppose that P, A, and B are as above and r_0 is the critical number for P and $\overrightarrow{AB}$. Let D be a point on the same side of $\overleftrightarrow{PA}$ as B so that $m(\angle APD) = r_0$. This angle, $\angle APD$ is called the* angle of parallelism *for P and $\overrightarrow{AB}$.*

Note that since there are two sides of $\overleftrightarrow{AP}$ there are really two angles of parallelism for P and the line $\overleftrightarrow{AB}$. There is a left-hand angle of parallelism and a right-hand angle of parallelism.

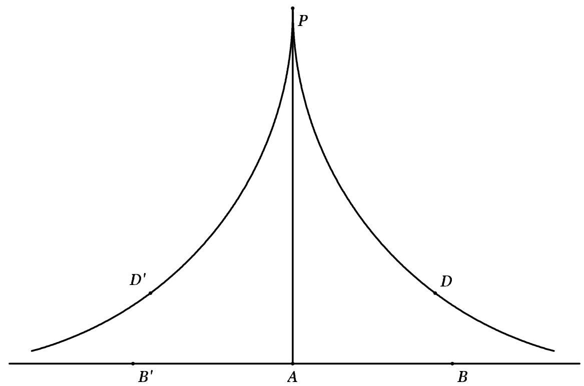
Are these two angles equal? Should they be?

Theorem 7.11 *The critical number depends only on $d(P, \ell)$.*

PROOF: Let ℓ be a line and $P \notin \ell$. Let A be the foot of the perpendicular from P to ℓ, and let $B \in \ell$, $B \neq A$. Now, let P', ℓ', A', and B' be another such set up so that $PA \cong P'A'$. Now, we need to show that the critical number for P and $\overrightarrow{AB}$ is the same as the critical number for P' and $\overrightarrow{A'B'}$. We can do this by showing that the intersecting sets, K and K', are the same, since if $K = K'$ then they must have the same least upper bound.

Suppose that $r \in K$. Then $\overrightarrow{PD_r}$ intersects $\overrightarrow{AB}$ at a point $T \in \overrightarrow{AB}$. Choose $T' \in \overrightarrow{A'B'}$ so that $AT \cong A'T'$. Then $\triangle PAT \cong \triangle P'A'T'$ by SAS, therefore $r \in K'$ and $K \subseteq K'$.

Similarly, we can show that if $s \in K'$ then $s \in K$ making $K' \subseteq K$. Therefore, $K = K'$. ∎



Now, we have seen that the critical number depends only on $d(P, \ell)$, we can think of it as a function from the reals to the reals.

Definition 7.3 *Given $x \in \mathbb{R}$, $x > 0$, locate a point P and line ℓ such that $d(P, \ell) = x$. Then define $\kappa(x)$ to be the critical number associated with P and ℓ, that is choose a point $B \neq A$ on ℓ and define $\kappa(x) = m(\angle APD)$, where $\angle APD$ is the angle of parallelism for P and $\overrightarrow{AB}$. By our above theorem, this is a well-defined function of x and does not depend on the choice of P, ℓ, or B. The function $\kappa\colon (0, \infty) \to (0, 90]$ is called the* critical function.

What happens to the size of the angle of parallelism as the point moves away from the line?

Theorem 7.12 *The function $\kappa\colon (0, \infty) \to (0, 90]$ is a decreasing function.*

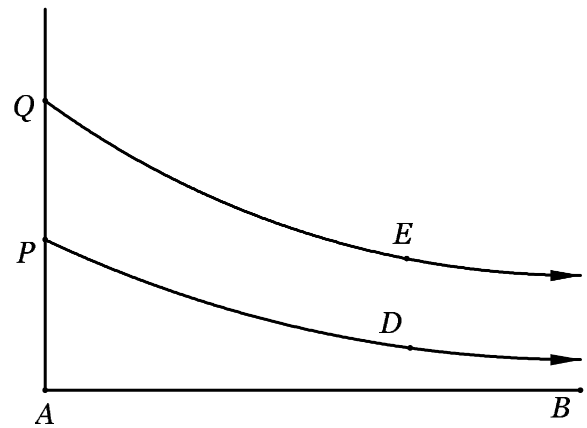
Proof: Let a and b be positive numbers so that $a < b$. We need to show that $\kappa(a) \geq \kappa(b)$. Choose P, A, B, and D so that $d(P, A) = a$ and $\angle APD$ is the angle of parallelism for P and $\overrightarrow{AB}$. Now, choose a point $Q \in \overrightarrow{AP}$ so that $d(A, Q) = b$. We have to show that the measure of the angle of parallelism at Q is no greater than the angle of parallelism at P.

Let $r = m(APD)$ and choose a point E on the same side of $\overleftrightarrow{AP}$ as B so that $m(\angle AQE) = r$. Thus, by the Corresponding Angles Theorem $\overleftrightarrow{QE} \parallel \overleftrightarrow{PD}$. Then since all of the points of $\overleftrightarrow{QE}$ are on one side of $\overleftrightarrow{PD}$ and all the points of $\overrightarrow{AB}$ are on the other, so $\overrightarrow{QE} \cap \overrightarrow{AB} = \emptyset$. This means that $r \notin K(Q, \overrightarrow{AB}$, so r cannot be less than the critical number for Q and $\overrightarrow{AB}$. This means that r, which is the measure of the angle of parallelism at P, must be greater than or equal to the angle of parallelism at Q. ∎

In hyperbolic geometry, this angle of parallelism is an extremely important concept. Note that in Euclidean geometry, the angle of parallelism for every situation is 90, so it would not seem to be an interesting concept. In hyperbolic geometry the angle of parallelism is an acute angle ***and*** every acute angle is the angle of parallelism for some situation.

Theorem 7.13 *Every angle of parallelism is acute and every critical number is less than 90.*

Proof: Let ℓ be a line and $P \notin \ell$. Drop a perpendicular from P to $A \in \ell$ and call this line m. We have shown that we can construct a perpendicular line to m at P, call it n, and $n \parallel \ell$. In hyperbolic geometry there must be at least one more line through P, call it t, $t \neq n$, that does not intersect ℓ. Since t and n both pass through P, and $n \perp m$, the other



line is not perpendicular to m, so the angle that it makes with $\overrightarrow{PA}$ on one side of m is less than 90. Now, the measure of this angle between t and $\overrightarrow{PA}$ is not in the intersecting set, since the line is parallel to ℓ. Thus, the critical number cannot be larger than that measure. Thus, the critical number is less than 90 and the angle of parallelism is acute. ∎

This ray that gives us the angle of parallelism is called the *limiting parallel ray* for $\overrightarrow{AB}$.

One very nice property of parallel lines from Euclidean geometry is that if ℓ, m, and n are three distinct lines and $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$. In other words, the property of being parallel was a transitive property. Clearly, in hyperbolic geometry we have given up that property as the universal hyperbolic axiom gives us an example that voids this. However, is some of this idea salvageable? Might it be possible that if we have three limiting parallel rays that are parallel to each other in the same direction, then some type of transitivity might be true?

First, we need to see how well-behaved these limiting parallel rays are. If we have that $\overrightarrow{PQ}$ is limiting parallel to $\overrightarrow{AB}$, does every ray that is contained in $\overrightarrow{PQ}$ have to be limiting parallel to $\overrightarrow{AB}$? It would seem reasonable, though the concept of limiting parallel seems to be tied to the foot of a point in the line. Our next theorem tells us that this property is true.

We state the following theorems without proof.

Theorem 7.14 *If $\overrightarrow{PQ}$ is limiting parallel to $\overrightarrow{AB}$, then $\overrightarrow{AB}$ lies in the interior of the angle $\angle APQ$.*

Theorem 7.15 *Let $P \notin \ell$ and let B lie between A and C in ℓ. Then $\overrightarrow{PQ}$ is limiting parallel to $\overrightarrow{AB}$* ***if and only if*** *$\overrightarrow{PQ}$ is limiting parallel to $\overrightarrow{BC}$.*

Theorem 7.16 *Let B lie between A and C on ℓ. $\overrightarrow{AB}$ is limiting parallel to $\overrightarrow{DE}$* ***if and only if*** *$\overrightarrow{BC}$ is limiting parallel to $\overrightarrow{DE}$.*

Theorem 7.17 *If $\overrightarrow{PQ}$ is limiting parallel to $\overrightarrow{AB}$, then $\overrightarrow{AB}$ is limiting to $\overrightarrow{PQ}$.*

What this means is that the rays $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ can be extended into lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ which are parallel.

Theorem 7.18 *If $\overrightarrow{PQ}$ is limiting parallel to $\overrightarrow{AB}$, then $\overleftrightarrow{PQ} \parallel \overleftrightarrow{AB}$.*

With this, we can make the following definition.

Definition 7.4 *A ray* $\overrightarrow{PQ}$ *is* parallel *to a line* ℓ *if* $\overrightarrow{PQ}$ *is a limiting parallel ray to some ray in* ℓ*. If this is the case then we say that the line* k *containing the ray* $\overrightarrow{PQ}$ *is* limiting parallel *to the line* ℓ *in the direction of* $\overrightarrow{PS}$*. These lines are also called* asymptotically parallel *or* horoparallel*.*

We have just proven that this parallelism is symmetric and we may denote them by

$$\overrightarrow{PQ} \parallel \ell \qquad \ell \parallel \overrightarrow{PQ} \qquad \ell \parallel k \qquad k \parallel \ell.$$

If $k = \overleftrightarrow{AB}$, $\ell = \overleftrightarrow{CD}$, and $\overrightarrow{AB} \parallel \overrightarrow{CD}$, then $\overrightarrow{AB}$ is said to be parallel to ℓ in the direction of $\overrightarrow{CD}$ on ℓ. Furthermore, k and ℓ are said to be parallel in the direction of $\overrightarrow{AB}$ on k and in the direction of $\overrightarrow{CD}$ on ℓ.

Theorem 7.19 *IF* $P \notin \ell$ *then there are exactly two lines through* P *that are limiting parallel to* ℓ*. Each contains an arm of the fan angle* $\angle(P, \ell)$ *and they are limiting parallel to* ℓ *in opposite directions.*

Theorem 7.20 (Weak Transitivity of Parallels) *Two lines parallel to a third* ***in the same direction*** *on the third are parallel to each other.*

7.7 Classification of Parallels

As we have mentioned we have seen two different properties of parallel lines. We have seen that there are limiting parallel lines and we have seen that there are parallel lines that admit a common perpendicular. Are these the same?

We will state the following results without proof, but the proof can be found in the references listed at the end.

Theorem 7.21 (Classification of Parallels, I) *If* ℓ *and* m *are limiting parallel lines, then* ℓ *and* m *do not admit a common perpendicular. If* ℓ *and* m *admit a common perpendicular, then they are not limiting parallel.*

Okay, then they are different, but how do they differ?

Theorem 7.22 *Suppose that* $\ell \parallel m$*. Let* P*,* Q*, and* R *be points on* m *with* $P * Q * R$ *and let* A*,* B*, and* C*, respectively, be the feet of the perpendiculars from* P*,* Q*, and* R *to* ℓ*.*

i) If $\overleftrightarrow{PA} \perp m$*, then* $PA < QB < RC$*.*

ii) If $\overrightarrow{PQ}$ *is limiting parallel to* $\overrightarrow{AB}$*, then* $PA > QB > RC$*.*

Thus, as we move in the "direction of parallelism" for limiting parallel lines, they tend to get closer together, while the other type of parallel lines tend to get further apart.

If $\ell \parallel m$ and ℓ and m admit a common perpendicular line, then the lines are said to be **hyperparallel**. This is denoted by $k\ )(\ \ell$. They are sometimes called *divergently parallel lines.*

This can be formalized in the following result.

Theorem 7.23 *IF ℓ and m are hyperparallel, then for every positive $r \in \mathbb{R}$ there is a point $P \in m$ so that $d(P, \ell) > r$. Furthermore, P may be chosen on either side of the common perpendicular.*

Now that we have two types of parallelism, is that all there is? Might there be another type of parallelism that we haven't yet studied? Fortunately, the answer is no.

Theorem 7.24 (Classification of Parallels, II) *If $\ell \parallel m$ then either ℓ and m admit a common perpendicular or ℓ and m are limiting parallel.*

While these are very important results, we will not need them in later sections, so we won't include proofs.