MA 113 — Calculus I Fall 2012
Exam 3 13 November 2012

Name: ______________________________

Section: __________

Last 4 digits of student ID #: __________

This exam has ten multiple choice questions (five points each) and five free response questions (ten points each). Additional blank sheets are available if necessary for scratch work. No books or notes may be used. Turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities.

On the multiple choice problems:

1. You must give your *final answers* in the *multiple choice answer box* on the front page of your exam.
2. Carefully check your answers. No credit will be given for answers other than those indicated on the *multiple choice answer box.*

On the free response problems:

1. Clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. Give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each free response question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Multiple Choice Answers

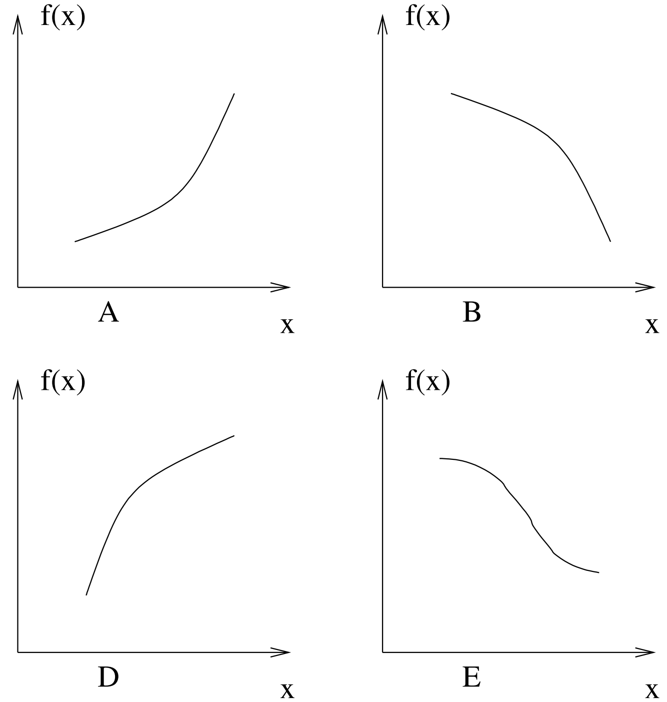
Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

Exam Scores

Question	Score	Total
MC		50
11		10
12		10
13		10
14		10
15		10
Total		100

Record the correct answer to the following problems on the front page of the exam.

1. The function f satisfies $f'(x) < 0$ for all x and $f''(x) < 0$ for all x. Which of the following could be the graph of f?



Option B. The function is decreasing and concave down.

2. Let the function f be defined by $f(x) = |\sin(x)|$. How many critical points does the function f have on the open interval $(\pi/4, 5\pi/4)$?

A. 0

B. 1

C. 2

D. 3

E. 4

Option C. There are two critical points, one at $\pi/2$ and one at π.

Record the correct answer to the following problems on the front page of the exam.

3. Let f be a function whose domain is the interval $(0, 2)$ and assume that f is differentiable on $(0, 2)$. Select the statement that must be true for any such f.

A. If f has a local maximum at 1, then $f'(1) = 0$.

B. If $f'(1) = 0$, then f has a local maximum at 1.

C. If $f(1) = 2012$, then f has a local maximum at 1.

D. If $f''(1) > 0$, then f has a local minimum at 1.

E. If $f(1/2) = f(3/2)$, then $f'(1) = 0$.

Option A. A local maximum occurs at a critical point. Since f is differentiable and $x = 1$ is a critical point, we must have $f'(1) = 0$.

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4. Suppose that f is a function whose domain is the open interval $(-1, 1)$ and f has a local maximum at 0. One of the statements below can never be true for f. The other four will be true for some choices of f and false for other choices. Which of the following can never be true for any such f?

A. $f'(0) = 0$ and $f''(0) < 0$

B. $f'(0) = 0$ and $f''(0) = 0$

C. $f'(0)$ does not exist

D. $f'(0) = 0$ and $f'(x) < 0$ for $0 < x < 1$

E. $f'(0) = 0$ and $f'(x) > 0$ for $0 < x < 1$

Option E. In option E, we have f increasing to the right of 0 and then 0 cannot be a local maximum.

Record the correct answer to the following problems on the front page of the exam.

5. Suppose that f is a function on the open interval $(0, 3)$ and we know the following information about the derivative f':

$$\begin{array}{ll} f'(x) > 0, & 0 < x < 1 \\ f'(1) = 0, & \\ f'(x) < 0, & 1 < x < 2 \\ f'(2) = 0, & \\ f'(x) < 0, & 2 < x < 3 \end{array}$$

Which of the following is true?

A. The function f has a local maximum at 1 and a local minimum at 2.

B. The function f has a local maximum at 1 and a local maximum at 2.

C. The function f has a local minimum at 1 and a local minimum at 2.

D. The function f has a local minimum at 1 and a local maximum at 2.

E. The function f has a local maximum at 1 and does not have an extremum at 2.

Option E. This follows from our first derivative test for increasing/decreasing behavior and for local extrema.

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6. We let $f(x) = x^2 - 8$ and use Newton's method to find a solution of $f(x) = 0$. If $x_1 = 4$, find the exact value of x_3.

A. $\sqrt{8}$

B. 8

C. 3

D. 17/6

E. None of the above.

Option D. We have $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - (x_n^2 - 8)/(2x_n) = (x_n + 8/x_n)/2$. Thus $x_2 = 3$ and $x_3 = (3 + 8/3)/2 = 17/6$.

Record the correct answer to the following problems on the front page of the exam.

7. Let $(a, f(a))$ be the point where the tangent line to the graph of $f(x) = x\cos(x)$ is horizontal. We use Newton's method to find $x_1, x_2, x_3, \ldots$, the successive approximations to a. Give the formula that we use to compute x_{n+1} from x_n.

A. $x_{n+1} = x_n - \dfrac{x_n \cos(x_n)}{\cos(x_n) - x_n \sin(x_n)}$

B. $x_{n+1} = x_n - \dfrac{\cos(x_n) - x_n \sin(x_n)}{x_n \cos(x_n)}$

C. $x_{n+1} = x_n + \dfrac{\cos(x_n)}{\sin(x_n)}$

D. $x_{n+1} = x_n + \dfrac{2\sin(x_n) + x_n \cos(x_n)}{\cos(x_n) - x_n \sin(x_n)}$

E. $x_{n+1} = x_n + \dfrac{\cos(x_n) - x_n \sin(x_n)}{2\sin(x_n) + x_n \cos(x_n)}$

Option E. To solve $f'(x) = 0$, use the iteration $x \to x - f'(x)/f''(x)$. We have $f(x) = x\cos(x)$, $f'(x) = \cos(x) - x\sin(x)$, and $f''(x) = -2\sin(x) - x\cos(x)$.

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8. Which of the following is **not** an anti-derivative of $2\sin(x)\cos(x)$?

A. $1 + \sin^2(x)$

B. $\sin^2(x)$

C. $1 - \cos^2(x)$

D. $-\cos^2(x)$

E. $\sin^2(x) + \cos^2(x)$

Option E. The derivative of $1 = \sin^2(x) + \cos^2(x)$ is 0 and $\sin(x)\cos(x)$ is not zero. Thus the function in E is not an antiderivative of $\sin(x)\cos(x)$.

Record the correct answer to the following problems on the front page of the exam.

9. We have a sequence of numbers $\{a_1, a_2, a_3, \dots\}$ so that $\displaystyle\sum_{k=1}^{n} a_k = n^2$ holds for $n = 1, 2, 3, \dots$.

Find the value of $\displaystyle\sum_{k=1}^{12}(4a_k + 2)$.

A. 578

B. 600

C. 2306

D. 2328

E. 2400

Option B. $\sum_{k=1}^{12}(4a_k + 2) = 4\sum_{k=1}^{12} a_k + \sum_{k=1}^{12} 2 = 4 \cdot 144 + 24 = 600.$

10. Let $f(x) = 3 - x$. Subdivide the interval $[1, 3]$ into four equal sub-intervals and compute R_4, the value of the right-endpoint approximation to the area under the graph of f on the interval $[1, 3]$.

A. 5/2

B. 2

C. 3/2

D. 1

E. None of the above.

Option C. The values at the right endpoints are 3/2, 1, 1/2, 0 and the approximate area is (1/2) (3/2 + 1 + 1/2 +0) = 3/2.

Free Response Questions: Show your work!

11. The volume of a sphere of radius r is $V(r) = 4\pi r^3/3$.

(a) Find $L(r)$, the linearization of the volume V at $r = 10$.

(b) A sphere has radius of 10 centimeters. The sphere is heated and the radius increases 6%. Use the linearization to approximate the increase in the volume of the sphere. Please give your answer as a multiple of π.

a) The linearization at 10 is $L(r) = V'(10)(r-10) + V(10)$ $= 400\pi(r-10) + \frac{4000\pi}{3}$.	Formula for linearization (2 points) compute V' (1 point) substituting $V(10)$ and $V'(10)$, (2 points)
b) The increase in the radius is 6% or 0.6 centimeters. The resulting increase in the volume is approximately $L(10.6) - L(10) = 0.6 \cdot 400\pi = 240\pi$ cubic centimeters.	increase in radius (2 points), increase in volume (2 points), units (1 point)) As a decimal, $240\pi \approx 753.98$, but the exact answer is preferred.

Free Response Questions: Show your work!

12. Let $f(x) = x^3 + 3x^2 - 9x + 2012$.

(a) Find the open intervals where f is increasing or decreasing.

(b) Find the intervals where f is concave up or concave down.

(c) Give all inflection points of f.

Use calculus to justify your answers.

Solution	Points
a) The first derivative is $f'(x) = 3x^2+6x-9 = 3(x+3)(x-1)$. We have the following information about the sign of f. x: $(-\infty, -3)$, $(-3, 1)$, $(1, \infty)$ $f'(x)$: positive, negative, positive Behavior of f: increasing, decreasing, increasing The function is increasing on the intervals $(-\infty, -3)$ and $(1, \infty)$. The function is decreasing on the open interval $(-3, 1)$	f' (1 point), $f' > 0$ for increasing and $f' < 0$ for decreasing (1 point), behavior on each interval (3x1 point),
b) The second derivative is $f''(x) = 6x-6$. We have $f''(x) > 0$ for x in $(-1, \infty)$ and $f''(x) < 0$ for x in $(-\infty, -1)$. Thus f is concave up on $(-1, \infty)$ and concave down on $(-\infty, -1)$.	f'' (1 point), $f'' > 0$ for concave up and $f'' < 0$ for concave down (1 point), each interval (2x1 point)
c) The point $(-1, f(1)) = (-1, 2023)$ is an inflection point since f changes concavity at $x = -1$.	1 point

13. Evaluate the following limits. Explain your reasoning.

(a) $\lim_{x\to 1} \dfrac{\ln(x)}{x^2-1}$.

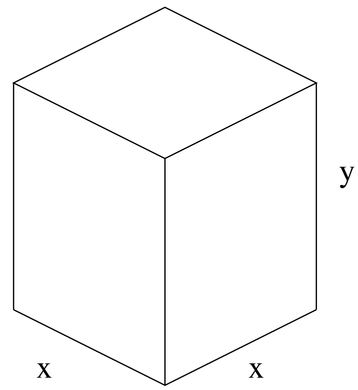
(b) $\lim_{x\to 1} \dfrac{e^x}{x^2-2}$

(c) $\lim_{x\to 0} \dfrac{1-\cos(3x)}{x^2}$

a) This limit is the indeterminate form 0/0 and we may use L'Hôpital's rule to evaluate the limit $\lim_{x\to 1}\frac{\ln(x)}{x^2-1} = \lim_{x\to 1}\frac{1/x}{2x} = \frac{1}{2}$.	Indeterminate form (1 point), apply L'Hôpital (1 point), value of limit (1 point)
b) This limit is not an indeterminate form. The function $f(x) = e^x/(x^2-2)$ is continuous at $x = 1$ and and by the definition of a continuous function, we may evaluate the limit by substituting $x = 1$. $$\lim_{x\to 1}\frac{e^x}{x^2-2} = -e.$$	Use of continuity or direct substitution rule (1 point), value of limit (2 point)
c) The limit is the indeterminate form 0/0. Applying L'Hôpital's rule gives another limit that is the indeterminate form 0/0. A second application of l'Hôpital's rule gives that the value of the limit is 9/2. $$\lim_{x\to 0}\frac{1-\cos(3x)}{x^2} = \lim_{x\to 0}\frac{3\sin(3x)}{2x} = \lim_{x\to 0}\frac{9\cos(3x)}{2}.$$	Indeterminate form (1 point), each application of L'Hôpital (1 point), answer (1 point)

Free Response Questions: Show your work!

14. A large shipping box has a square base of sidelength x meters. The height of the container is y meters. Central Associated Transport Services (CATS) will only accept the box if the sum of the height and the perimeter of the base is equal to 10 meters.



(a) Write down a function which gives the volume of the box in terms of x, the sidelength of the base. Give the domain of the function which gives the volume.

(b) Find the dimensions x and y of the box with the **largest** possible volume.

(c) Explain how we know we have found the largest possible volume.

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a) The volume of the box is $V = x^2y$. We have the restriction that $4x + y = 10$. Solving for y and substituting gives $V(x) = 10x^2 - 4x^3$. Since x and y are sidelengths, they must be positive (we allow zero). This gives $x \geq 0$ and $10 - 4x \geq 0$ or $x \leq 5/2$. Thus, we consider V on the domain $0 \leq x \leq 5/2$.

Constraint $4x + y = 10$ (1 point), Volume is x^2y (1 point), eliminate x (1 point), domain (1 point)).

b) To find the maximum, we need to list the critical points and the endpoints. We compute $V'(x) = 20x - 12x^2 = 4x(5 - 3x)$. We solve $V'(x) = 0$ to find the critical point is $5/3$ and the endpoints are 0 and $5/2$. Testing values we obtain

x	0	$5/3$	$5/2$
$V(x)$	0	$250/27$	0

The largest volume is $250/27$ cubic meters (or approximately 9.3 cubic meters.) The volume occurs for a box with $x = 5/3$ meters and $y = 10/3$ meters.
Do not deduct if 0 is listed as a critical point, also.

Critical point (1 point), endpoints (1 point), list of volumes (1 point), dimensions of largest box (2 points)

c) We know that a continuous function a closed interval has a largest value and it must occur at a critical point or endpoint.
Students may argue that the largest value occurs when $x = 5/3$ since V is increasing on [0,5/3] and decreasing on [5/3,5/2]. Accept this.
The statement $V''(5/3) < 0$ only implies V has a local maximum at $5/3$, not an absolute maximum.

(1 point)

Free Response Questions: Show your work!

15. Let g be a twice differentiable function and suppose that $g''(x) = 3x + \cos(x)$. If $g'(0) = 2$ and $g(1) = 3$, find the function g.

Taking anti-derivatives twice, we find that $g'(x) = 3x^2/2 + \sin(x) + A$ and $g(x) = x^3/2 - \cos(x) + Ax + B$.	Anti-derivatives (2x3 points),
The conditions $g'(0) = 2$ gives that $A = 2$ and then $g(1) = 1/2 - \cos(1) + 2 + B = 3$ gives that $B = 1/2 + \cos(1)$.	equations for constants (2 points), values of constants (2 points). As a decimal, $\frac{1}{2} + \cos(1) \approx 1.0403$. The exact answer is preferred.
Thus $$g(x) = \frac{x^3}{2} - \cos(x) + 2x + \frac{1}{2} + \cos(1).$$	