$\frac{24}{24}$

Ma 322
Written Homework 2 Feb 2, 2017

① ⓐ A 6x6 matrix with 6 pivot points forces the matrix below. With this the augmented matrix can also be found for $Ax=0$

$$\begin{bmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{bmatrix} = A \qquad \left[\begin{array}{cccccc|c} 1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0 \end{array}\right]$$

$\frac{4}{4}$

These matrices can be verified by observation since any row in A could be used to get rid of all values to the right of the pivot variables. Now that we have the augmented matrix we easily see by observation that this system only has the trivial solution.

ⓑ
$$Ax=b \rightarrow \begin{bmatrix} 1&0&0&0&0&0&b_1\\0&1&0&0&0&0&b_2\\0&0&1&0&0&0&b_3\\0&0&0&1&0&0&b_4\\0&0&0&0&1&0&b_5\\0&0&0&0&0&1&b_6 \end{bmatrix}$$

With this matrix it is easy to see that for every b vector there is always the solution of 6. Therefor, there is always at least one solution for each b.

② ⓐ A 6x6 matrix with 2 pivot points has exactly 2 basic variables. This means there are exactly 4 free variables. Since there is at least one free variable this means there is a at least one non trivial solution.

Ex) $\left[\begin{array}{cccccc|c} 1&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{array}\right] \rightarrow \begin{cases} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3, x_4, x_5, x_6 - \text{free} \end{cases}$

4/4

ⓑ $Ax = b$ for every possible b is not guarenteed to have any solution. Easy to see with following example.

$\left[\begin{array}{cccccc|c} 1&0&0&1&0&0&0 \\ 0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0 \end{array}\right]$ contratiction $0 \neq 1$

③ ⓒ Using similar logic as in #1 the matrix A is forced to always have 0s to the right of each pivot column in rref.

$\begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} = A \qquad Ax = 0 \rightarrow \left[\begin{array}{ccccc|c} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{array}\right]$

When in this form it is easy to see that there is only the trivial solution.

(6)

$Ax = b \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 1 & 0 & b_4 \\ 0 & 0 & 0 & 0 & 1 & b_5 \\ 0 & 0 & 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & 0 & 0 & b_7 \end{array}\right]$ This implies $0 = b_7$ which is only true when $b_7 = 0$. This restricts the number of valid b possibilities. Therefor, there is not a solution for every b.

4/4

(4) (a) A 5×7 matrix with five pivot points will have exactly two free variables. Since there are above 0 free variables, it will have more than the trivial solution. Same logic as previous example in #2.

(b) $Ax = b$ will always have a solution because it is impossible to form a contradiction since the left hand side of the augmented matrix will always have at least one non zero and there is no real restriction on the right hand side.

4/4

(5)

$$\text{Span}\left\{\begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-5\\3\end{bmatrix}\right\} \text{ for } x_1 + 3x_2 + 5x_3 = 0$$

4/4

$$\begin{bmatrix}7\\0\\0\end{bmatrix} + \text{Span}\left\{\begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-5\\3\end{bmatrix}\right\} \text{ for } x_1 + 3x_2 + 5x_3 = 0$$

These two solutions are very similar just the second equation can't be represented with just a simple span. But by taking a simple solution and adding the span we can calculate all solutions.

Note for #5) we know that it only takes two vectors to represent the span since the solutions are in $\mathbb{R}^3$. So, if any other vectors were in the span it would be redundant since it would be a linear combination of the first two.

(6) for a 5×3 matrix you will need ~~at least~~ 3 pivot columns. If any lower

(YOU ALSO CAN'T HAVE MORE)

than this assume a matrix where all elements are zero except the pivot colums. It simple to see how given three colums $C_1, C_2, \& C_3$ that $0 \cdot C_1 + 0C_2 = C_3$. However with 3 pivot colums you can ensure that $0 \cdot C_1 + 0 \cdot C_2 \neq C_3$.

$\frac{4}{4}$

A 3×5 matrix can never have 5 linearly independent colums. Since all collumns are in $\mathbb{R}^3$ and it is easy to see how three pivot collumns span all of $\mathbb{R}^3$. There are no available linearly independent vectors in $\mathbb{R}^3$ to chose for the last two colums. Conversly, the logic used in the first part of #6 shows why the only <u>possible</u> valid pivot column number is 3.