

Written Assignment 6 for MA 322 - Matrix Algebra (Spring 2017)

March 30, 2017

Problem 1

Suppose A is a 5×5 matrix with exactly three eigenvalues, and we know one of the eigenspaces is three-dimensional (we aren't told about the others). Can we determine whether or not A is diagonalizable? Explain why or why not.

If A is a 5×5 matrix with exactly three eigenvalues, then we know that there are exactly three eigenspaces, each of which is at least one-dimensional. We also know that the sum of the dimensions of all eigenspaces is less than or equal to the number of columns in the original matrix. If one eigenspace is known to be three-dimensional, then in order for the dimensions of all three eigenspaces to add up to 5 we know that each of the other two eigenspaces must be exactly one-dimensional. Since we know that each eigenspace is at least one-dimensional without having to verify it using the exact eigenvectors, we can immediately determine that A is diagonalizable.

Problem 2

Now suppose A is a 6×6 matrix with exactly three eigenvalues, and one of the eigenspaces is three-dimensional (we aren't told about the others). Can we determine whether or not A is diagonalizable? Explain why or why not.

This time, in order for the dimensions of all three eigenspaces to add up to 6, one of the two remaining eigenspaces must be one-dimensional, and the other two-dimensional. Since the existence of a two-dimensional eigenspace cannot be verified without checking the eigenvectors, we $\boxed{\text{cannot}}$ determine whether or not A is diagonalizable without more information.

Problem 3

Construct an example of a matrix that is diagonalizable, is not diagonal, and is not invertible.

In order for a matrix to be diagonalizable, it must have exactly as many linearly independent eigenvectors as columns. In order for the matrix to also not be invertible, it must have linearly dependent rows, which always happens whenever one of the eigenvalues is 0. Therefore, one example of a diagonalizable (yet not diagonal) matrix with 0 as one of the eigenvalues is a 2×2 matrix with eigenvalues 0 and 1, or characteristic equation $\lambda^2 - \lambda = 0$. Creating a matrix with this equation is relatively easy - here is one such example:

$$\begin{bmatrix}
\frac{1}{2} & 1 \\
\frac{1}{4} & \frac{1}{2}
\end{bmatrix}$$
 (characteristic equation = $\left(\lambda^2 - \lambda + \frac{1}{4}\right) - \left(\frac{1}{4}\right) = 0 \Leftrightarrow \lambda^2 - \lambda = 0$)

Problem 4

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, and $\mathscr{B} = \{b_1, b_2\}$ where $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by T(x) = Ax. Find $[T]_{\mathscr{B}}$.

Step 1: Figure out the linear transformation of each basis vector:

$$T(\mathbf{b}_1) = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 \\ -1 \cdot 1 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T(\mathbf{b}_2) = \begin{bmatrix} 1 \cdot 5 + 1 \cdot 4 \\ -1 \cdot 5 + 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$$

Step 2: Write these vectors with respect to \mathscr{B} , and make them the columns of $[T]_{\mathscr{B}}$:

$$[T(\mathbf{b}_1)]_{\mathscr{B}} = \text{solution to } \begin{bmatrix} 1 & 5 & | & 2 \\ 1 & 4 & | & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & | & 2 \\ 0 & -1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & | & 2 \\ 0 & 1 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$[T(\mathbf{b_2})]_{\mathscr{B}} = \text{solution to } \begin{bmatrix} 1 & 5 & | & 9 \\ 1 & 4 & | & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & | & 9 \\ 0 & -1 & | & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & | & 9 \\ 0 & 1 & | & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} [T]_{\mathscr{B}} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Problem 5

Let $A = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}$. The transformation $x \mapsto Ax$ is a composition of a rotation and a scaling.

a. Give the angle of rotation and the scale factor.

Scale factor (s) = norm of every column vector in A:

$$s = ||\mathbf{a}_1|| = \sqrt{(\sqrt{3})^2 + (-3)^2} = \sqrt{12} = 2\sqrt{3}$$

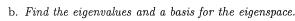
Dividing A by s makes the rotation easier to see:

$$A = 2\sqrt{3} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Since the rotation matrix is $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, $\cos \theta > 0$ and $\sin \theta < 0$, putting θ in Quadrant IV. Applying the appropriate reference angle, we get

$$A = 2\sqrt{3} \begin{bmatrix} \cos -\frac{\pi}{3} & -\sin -\frac{\pi}{3} \\ \sin -\frac{\pi}{3} & \cos -\frac{\pi}{3} \end{bmatrix}$$

which means that $\theta = -\frac{\pi}{3}$



Because this is a rotation, the eigenvalues are going to be imaginary:

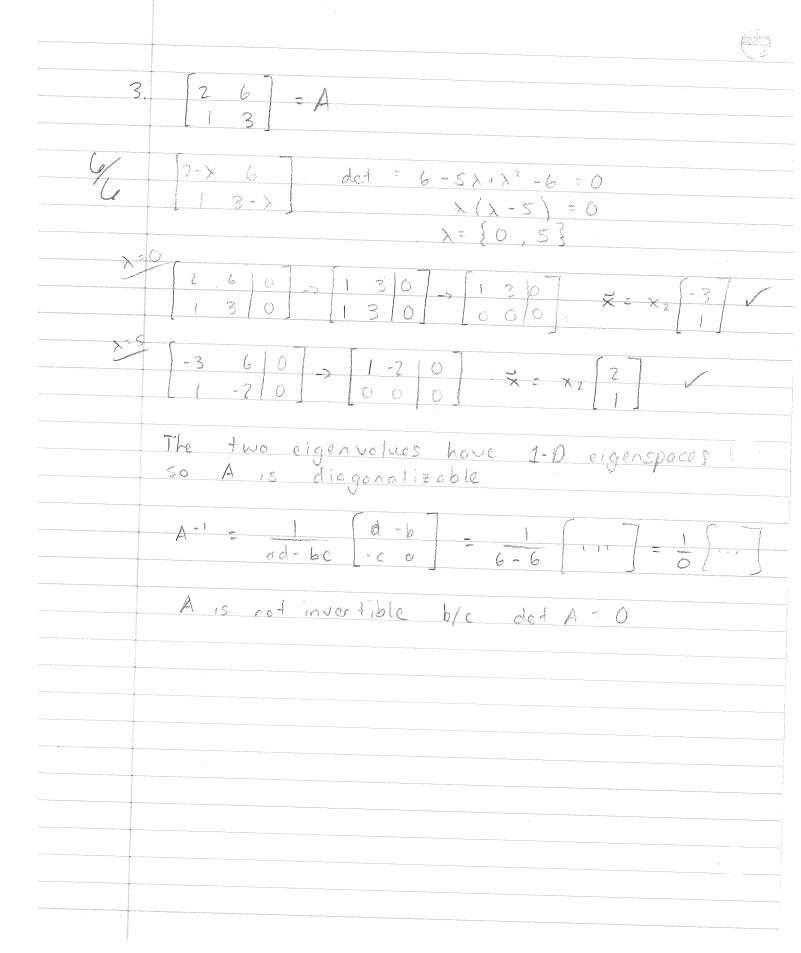
$$A - \lambda I = 0 \Leftrightarrow \begin{vmatrix} \sqrt{3} - \lambda & 3 \\ -3 & \sqrt{3} - \lambda \end{vmatrix} = 0 \Leftrightarrow (\sqrt{3} - \lambda)^2 + (3)^2 = 0$$
$$\Leftrightarrow 3 - 2\sqrt{3}\lambda + \lambda^2 + 9 = 0 \Leftrightarrow \lambda^2 - 2\sqrt{3}\lambda + 12 = 0$$
$$\Rightarrow \lambda = \frac{2\sqrt{3} \pm \sqrt{12 - 48}}{2} = \sqrt{3} \pm 3i$$

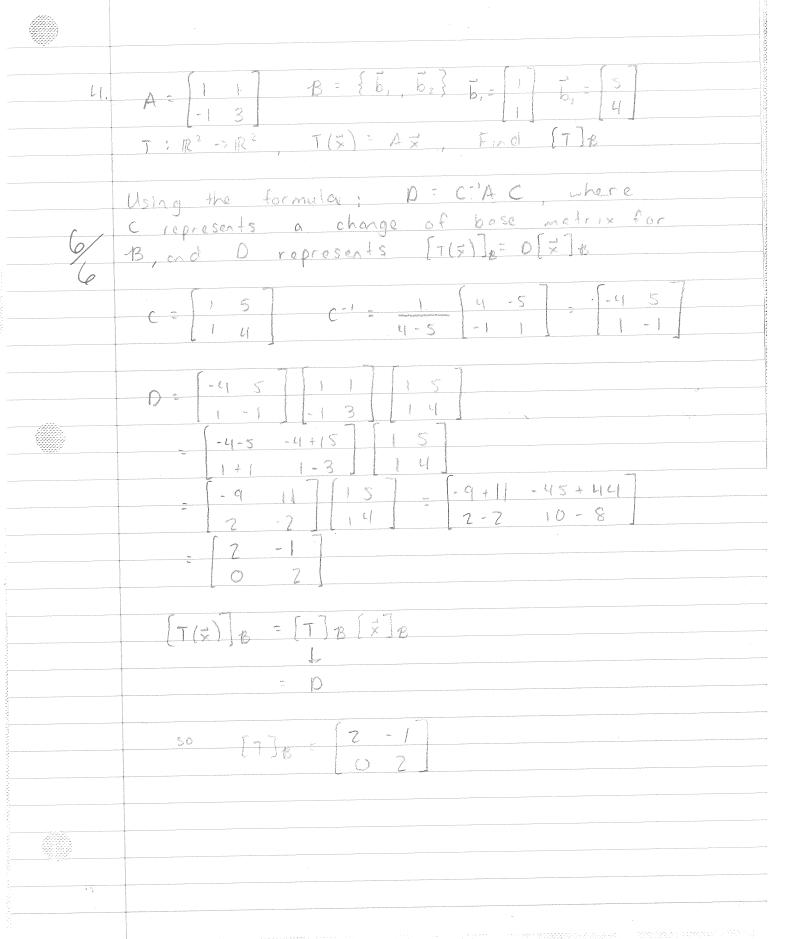
The eigenvectors can be computed in a very straightforward manner (NOTE: Only the computation for $\lambda = \sqrt{3} + 3i$ is shown - since A contains only real numbers, the other eigenvector must be the conjugate of this one.)

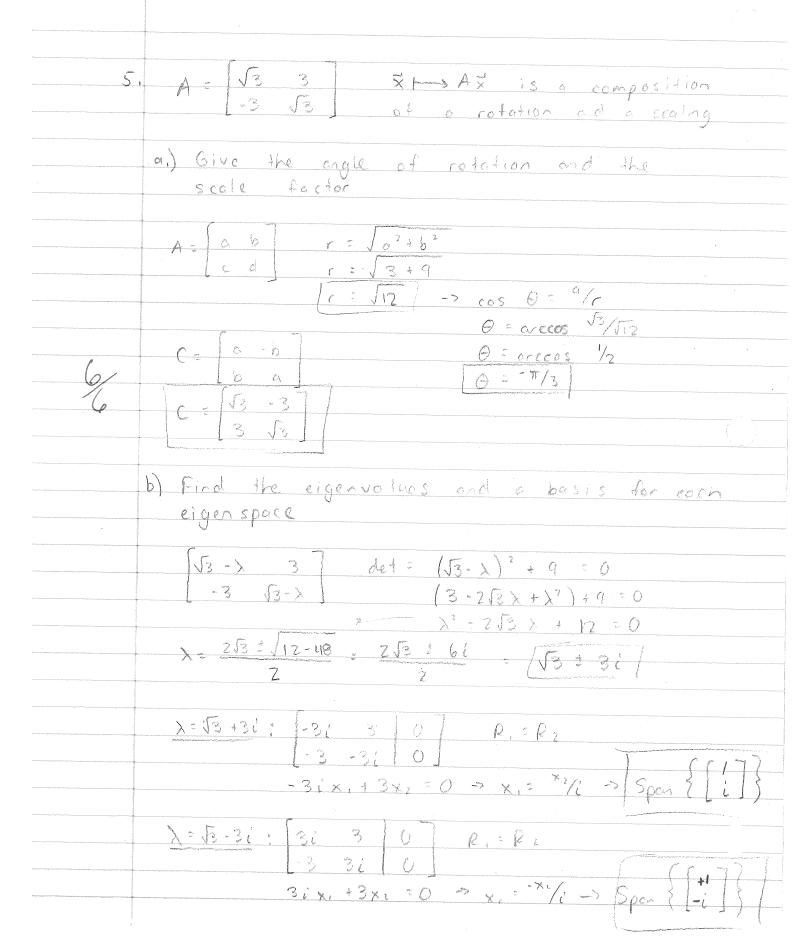
$$A - \lambda I = \begin{bmatrix} \sqrt{3} - \sqrt{3} - 3i & 3 \\ -3 & \sqrt{3} - \sqrt{3} - 3i \end{bmatrix} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} -3i & 3 \\ -3i & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -3i & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$
$$\Rightarrow x_1 = -ix_2 \Rightarrow \text{Nul } A = \text{Span} \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenvector for $\lambda_1 = \sqrt{3} + 3i$ is $\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$, and the eigenvector for $\lambda_1 = \sqrt{3} - 3i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Any matrix involving only rotation and scaling has these two eigenvectors. Therefore, the basis for the eigenspace is $\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

	Written HW 6 3/30/17
	A is 5 x 5, 3 eigenvalues, one eigenspace is three dimensional; can we determine if A is diagonalizable?
3	A will only be diagonalizable if it has s linearly independent eigenvectors. A only has three eigenvalues, but as long as the multiplicaties of these eigenvalues adds to five, A will be diagonalizable, since one eigenvalue has a three-dimensional ergenspace, and we can assume the other two have one-dimensional eigenspaces, leading to five linearly independent eigenvectors, making A diagonalizable
· .	is three-dimensional; can be determine if A is diagonalizable?
	Using the some reasoning as in \$1 if the other two eigenvalues only have one eigenvector apiece, there won't be enough eigenvectors to make A diagonalizable, So we can not determine if it is or







Written Honswork 6

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A three-dimensional eigenspace means that one of the eigenvalues produced 3 linearly independent eigenvectors. By definition, we know that the other two eigenvolves each produce at least I eigenvector in their eigenspace, otherwise they howldn't be eigenvalues. This brings the total eigenvectors up to 5. For a 5x5 matrix, we must have 5 linearly independent eigenvectors. As long as those 5 are linearly independent. A 15 diagonalizable.

and one of the eigenspece is three-dimensional (we oren't told about the others). Can be determine whether or not A is diagonalizable? Explain why or why not.

A three-dimensional eigenspace means 3 linearly independent eigenvectors. By definition, he know that the other two eigenvalues produce at least 1 eigenvector, otherwise they would not be eigenvalues. This brings the minimum up to 5 eigenvectors. For a loke matrix, he need to eigenvectors to be diagonalizable; therefore, one of those other eigenvalues reeds to have an eigenspace that is two-dimensional. If is, A is diagonalizable. If both of the other eigenvalues are one-dimensional.

Zz

3/2

Diegoralizable means that am nxn men'x must produce eigenvolves whose eigensposes have directions, when summed equal n. Not diagonal means that it cannot be both upper triangular and laker triangular at the same time. Not invertible means that the determinant must be o.

6 det(A-ZI) = (a-Z)(d-Z) -bc = 0 22-02-02 +ad=620=0 $\lambda^2 - \alpha \lambda - d\lambda = 0$ 2 (2-0-0) = 0

> $\lambda_1 = 0$ $\lambda_2 = 0 + d \neq 0$ 3 & 2 distinct eigenvolve meas 2 Numby independent eigen vectors which means diagonalitable.

b = 0 and c = 0 at some time. I not diagonal

Given these constraints above, a possible construction is:

$$A = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$$

proven by: detA = 6(2) - 2(4) = 0 + not invertible de+(A-2x)= (6-2)(2-2) -4(3)=0 7(2-8)=0 $\lambda_1 = 0$ $\frac{1}{2}$ and diagonalizable.

b=4, c=3 at not diegonol.

$$B[X]_{S} = T(G)$$

$$E[X]_{S} = G^{*}T(G)$$

$$G^{*} = [X, Y]$$

$$\begin{bmatrix} \vec{X}_1 \end{bmatrix}_8 = \begin{bmatrix} +5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$
Thus, form colons of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} \vec{X}_2 \end{bmatrix}_8 = \begin{bmatrix} -45 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$[T]_6 = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

(S) Let A = [3 3]. The transformation \$\frac{1}{2} \to A \frac{1}{2} is a composition of rotation and scaling.

(a) give the angle of rotofier and scale factor.

pulling rout of A:

$$R = \begin{bmatrix} 2\sqrt{3} & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 2\sqrt{3} & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$\cos \theta = \frac{1}{2}$$
 rotation = -60°
 $\theta = 60^\circ$ Scale = 2.55

(b) Find the eigenvalues and a basis for each eigenspace.

eiguspace for 7 = J3 + 3 i:

PI has form's
$$i \times i + \times i = 0$$

 $x_i = -\frac{x_2}{i}$ let $x_2 = i$
then $x_1 = -1$

basis for eigenspace: [ti]

eigenpae for
$$\lambda_{z}=\sqrt{3}-3i$$
:

$$A-\lambda I=\begin{bmatrix} -3i & 3\\ -3 & -3i \end{bmatrix} \rightarrow Augmented \rightarrow \begin{bmatrix} -i & i & 0\\ -1 & -i & 0 \end{bmatrix}$$

Pr has form:
$$-ix$$
, $+x_2=0$
 $x_1=\frac{x_2}{L}$ let $x_2=i$
then $x_1=1$