1. Given the differential equation $y'' - 2xy' + y = 0$, assume the solution can be written in the form of the series, and find the recurrence relation for the coefficients. Then find the first three non-zero terms of two independent solutions (y_1 and y_2).

Assume $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution to the differential equation.

Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$ since this is 0 for $n=0$ and

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$ (change of indices).

Substituting into the eqn.,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \underbrace{- 2x \sum_{n=0}^{\infty} n a_n x^{n-1}}_{= -2 \sum_{n=0}^{\infty} n a_n x^n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2n a_n + a_n \right] x^n = 0.$$

So, $(n+2)(n+1) a_{n+2} - 2n a_n + a_n = 0$

$\Rightarrow \boxed{(n+2)(n+1) a_{n+2} + (1-2n) a_n = 0}$, and this gives us a recurrence relation for the coefficients.

Now, we compute the first few coefficients: (this will help us below!)

- $n=0$: $2a_2 + a_0 = 0$, so $a_2 = \frac{-a_0}{2}$
- $n=1$: $(3)(2) a_3 + (1-2) a_1 = 0 \Rightarrow 6a_3 - a_1 = 0 \Rightarrow a_3 = \frac{a_1}{6}$.
- $n=2$: $(4)(3) a_4 + (1-4) a_2 = 0 \Rightarrow 12 a_4 - 3 a_2 = 0 \Rightarrow a_4 = \frac{a_2}{4} = \left(\frac{-a_0}{2}\right)\left(\frac{1}{4}\right) = \frac{-a_0}{8}$
- $n=3$: $(5)(4) a_5 + (1-6) a_3 = 0 \Rightarrow 20a_5 - 5a_3 = 0 \Rightarrow a_5 = \frac{a_3}{4} = \left(\frac{a_1}{6}\right)\left(\frac{1}{4}\right) = \frac{a_1}{24}$.

From our definition, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$.

Note the coefficients depend on choice of a_0 and a_1.

If we let $a_0 = 1, a_1 = 0$, then we obtain

$$y_1 = 1 + 0x - \frac{1}{2}x^2 + 0x^3 - \frac{1}{8}x^4 + \cdots = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots$$

and if we let $a_0 = 0, a_1 = 1$, we obtain

$$y_2 = 0 + x + 0x^2 + \frac{1}{6}x^3 + 0x^4 + \frac{1}{24}x^5 + \cdots = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \cdots$$

2. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$, and an open interval where the series will converge absolutely. (Do not worry about the endpoints of the interval.)

To find the radius of convergence, we use the ratio test.

Note $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}(x+2)^n = \sum_{n=1}^{\infty} a_n (x+2)^n$ if $a_n = \frac{(-1)^n n^2}{3^n}$.

So, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^2} \right|$

absolute values mean we can ignore -1's

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \cdot \frac{1}{3} \right|$

$= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{3n^2} \right| = \frac{1}{3}$ (ratio of coefficients of highest powers)

So, since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = A$ the radius of convergence is $\underline{\frac{1}{A} = 3}$.

Therefore, the series will converge absolutely on the interval $(-2-3, -2+3) = (-5, 1)$.