SOLUTIONS

1. Given the differential equation $y'' - xy' - y = 0$, assume the solution can be written in the form of the series about $x_0 = 1$, and find the recurrence relation for the coefficients. Then find the first three non-zero terms of two independent solutions (y_1 and y_2).

note $x_0 = 1$ is an ordinary pt.

Assume $y = \sum_{n=0}^{\infty} a_n (x-1)^n$ is a solution to the differential eqn.

Then $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$.

Plugging in, $\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$

have to be careful w/ x in front

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - (x-1+1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$

match exponents

$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$

note making index 0 doesn't change anything!

combine

$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - (n+1) a_{n+1} - a_n \right] (x-1)^n = 0$

recurrence relation: $(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - (n+1) a_n = 0$

$\Rightarrow (n+2) a_{n+2} - a_{n+1} - a_n = 0$

$\Rightarrow \boxed{a_{n+2} = \dfrac{a_{n+1} + a_n}{n+2}}$

- $n=0$: $a_2 = \dfrac{a_1 + a_0}{2}$
- $n=1$: $a_3 = \dfrac{a_2 + a_1}{3} = \dfrac{\left(\frac{a_1 + a_0}{2}\right) + a_1}{3} = \dfrac{a_1 + a_0 + 2a_1}{2} \left(\frac{1}{3}\right)$

  $a_3 = \dfrac{3a_1 + a_0}{6} = \frac{1}{2} a_1 + \frac{1}{6} a_0$
- $n=2$: $a_4 = \dfrac{a_3 + a_2}{4}$

$= \left(\frac{1}{2} a_1 + \frac{1}{6} a_0 + \frac{1}{2} a_1 + \frac{1}{2} a_0\right) \frac{1}{4} = \left(\frac{2}{3} a_0 + a_1\right)\left(\frac{1}{4}\right) = \frac{1}{6} a_0 + \frac{1}{4} a_1$

- If $a_0 = 1$, $a_1 = 0$:

$y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$

- If $a_0 = 0$, $a_1 = 1$:

$y_2 = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots$

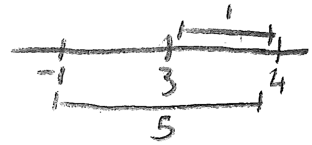
2. Determine a lower bound for the radius of convergence of a series solution about each x_0 for the differential equation $(x^2-2x-3)y''+xy'+4y=0$.

   a. $x_0=4$

   b. $x_0=-4$

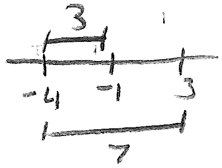
There is no common factor, so for $P(x)=x^2-2x-3$, we set $P(x)=0$.
Then $(x-3)(x+1)=0$, so $x=-1$, $x=3$.

(a)



A lower bound for the radius of convergence is 1, and series about $x_0=4$ will converge in $(4-1, 4+1)=(3,5)$.

(b)



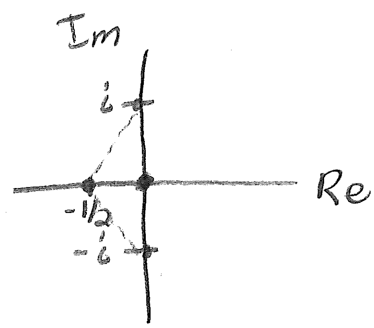
The lower bound is $\rho=3$, and the series about $x_0=-4$ will converge on $(-4-3, -4+3)=(-7,-1)$.

3. Determine a lower bound for the radius of convergence of a series solution about each x_0 for the differential equation $(1+x^2)y''+2xy'+4x^2y=0$.

   a. $x_0=0$

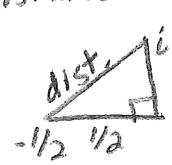
   b. $x_0=-\frac{1}{2}$

Again, there are no common factors.
$P(x)=1+x^2$.
$P(x)=0 \Rightarrow x^2=-1 \Rightarrow x=\pm i$.



(a) Both i and $-i$ are at distance 1 from $x_0=0$. Thus 1 is the lower bound for the radius of convergence and the series solution will converge for $|x|<1$.

(b) Distance from $-1/2$ to i is the same as from $-1/2$ to $-i$.



dist. $=\sqrt{(1/2)^2+1^2}=\sqrt{1/4+1}=\sqrt{5/4}=\boxed{\frac{\sqrt{5}}{2}}$ ← lower bound for radius of convergence.

The series solution will converge in $(-1/2-\sqrt{5}/2, -1/2+\sqrt{5}/2)$.