SOLUTIONS

1. Find the Laplace transform of $\int_0^t (t-w)^2 \cos 2w\, dw$.

   Hint: use the fact that $L(f*g) = F(s)G(s)$.

   If $f(t) = t^2$ and $g(t) = \cos 2t$, then $f*g = \int_0^t (t-w)^2 \cos 2w\, dw$.

   Using Laplace transform table,

   $F(s) = \frac{2}{s^3}$ and $G(s) = \frac{s}{s^2+(2)^2} = \frac{s}{s^2+4}$

   So, $\mathcal{L}\{\int_0^t (t-w)^2 \cos 2w\, dw\} = \mathcal{L}\{f*g\} = F(s)G(s) = \frac{2s}{s^3(s^2+4)}$

   $$\boxed{= \frac{2}{s^2(s^2+4)}}$$

2. Find the inverse Laplace transform of each expression. Your answers should include an integral.

   a. $\frac{s}{(s+1)(s^2+4)} = \underbrace{\frac{1}{s+1}}_{F(s)} \cdot \underbrace{\frac{s}{s^2+4}}_{G(s)}$

   So, $f(t) = e^{-t}$ and $g(t) = \cos 2t$.

   Thus $\mathcal{L}^{-1}\{\frac{s}{(s+1)(s^2+4)}\} = \mathcal{L}^{-1}\{F(s)G(s)\} = f*g = \underline{\int_0^t e^{-(t-w)} \cos 2w\, dw}$

   b. $\frac{G(s)}{s^2+16} = G(s) \cdot \underbrace{\frac{1}{s^2+16}}_{F(s)}$

   Let $g(t) = \mathcal{L}^{-1}\{G(s)\}$. Note $F(s) = \frac{1}{s^2+4^2} = \frac{1}{4}\left(\frac{4}{s^2+4^2}\right)$,

   so $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{4}\sin(4t)$.

   Hence $\mathcal{L}^{-1}\{\frac{G(s)}{s^2+16}\} = \mathcal{L}^{-1}\{F(s)G(s)\} = f*g = \underline{\int_0^t \frac{1}{4}\sin(4(t-w))\, g(w)\, dw}$

3. Use Laplace transforms to solve the initial value problem
$y''+4y'+4y=g(t); \quad y(0)=2,\ y'(0)=-3.$

Taking the Laplace transform of both sides of the differential equation, we have

$$s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))+4Y(s)=G(s)$$

plug in initial conditions $\rightarrow$

$$=s^2Y(s)-2s+3+4sY(s)-8+4Y(s)=G(s)$$

where $G(s)=\mathcal{L}(g(t))$ and $Y(s)=\mathcal{L}(y(t))$

$$=(s^2+4s+4)Y(s)-2s-5=G(s)$$

So, $Y(s)(s+2)^2=2s+5+G(s)$

$$\Rightarrow Y(s)=\frac{2s+5}{(s+2)^2}+\frac{G(s)}{(s+2)^2}$$

$$=\frac{2s+4+1}{(s+2)^2}+\frac{G(s)}{(s+2)^2}$$

$$=\frac{2\cancel{(s+2)}}{(s+2)^{\cancel{2}}}+\frac{1}{(s+2)^2}+\frac{G(s)}{(s+2)^2}$$

$$=\frac{2}{s+2}+\frac{1}{(s+2)^2}+G(s)\underbrace{\left(\frac{1}{(s+2)^2}\right)}_{F(s)}$$

Note if $F(s)=\frac{1}{(s+2)^2}$, $F(s)=\frac{1}{(s-(-2))^2}$, so $\mathcal{L}^{-1}\{F(s)\}=te^{-2t}$.

Therefore, $y(t)=\mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}\{\frac{2}{s+2}\}+\mathcal{L}^{-1}\{\frac{1}{(s+2)^2}\}+\mathcal{L}^{-1}(\frac{1}{(s+2)^2}\cdot G(s)\}$

$$=\underbrace{2e^{-2t}+te^{-2t}+\int_0^t (t-w)e^{-2(t-w)}g(w)\,dw}$$