(1a) $y' = x + y, \quad y(0) = 1$

$y' - y = x$ is 1^{st} order linear. Integrating factor

$\mu(x) = e^{\int p(x)dx} = e^{\int -1 dx} = e^{-x}$ Apply it:

$$\underbrace{e^{-x}y' - e^{-x}y}_{} = e^{-x}x$$

this is exactly $\frac{d}{dx}[e^{-x}y]$, by design. Integrate both sides with respect to x:

$$e^{-x}y = \int xe^{-x}\,dx \qquad \begin{bmatrix} u = x & dv = e^{-x} \\ du = 1dx & v = -e^{-x} \end{bmatrix}$$

$$e^{-x}y = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C$$

Multiply all by e^{x}:

$y = -x - 1 + Ce^{x}$ ← general solution

$y(0) = 1 \Rightarrow \quad 1 = -1 + Ce^{0} \quad \underline{C = 2}$

Final answer: $\boxed{y = -x - 1 + 2e^{x}}$

(1b) $y' = xy - 3y, \quad y(1) = 1$. This equation is separable:

$$\int \frac{dy}{y} = \int (x-3)\,dx \Rightarrow \ln y = \frac{1}{2}x^2 - 3x + C$$

$$\Rightarrow y = e^{\frac{1}{2}x^2 - 3x + C}$$

$\Rightarrow y = c_1 e^{\frac{1}{2}x^2 - 3x}$ ← general solution

$y(1) = 1 \quad 1 = c_1 e^{\frac{1}{2} - 3} \Rightarrow 1 = c_1 e^{-5/2}$

$\Rightarrow c_1 = e^{5/2}$

Final answer: $\boxed{y = e^{5/2} e^{\frac{1}{2}x^2 - 3x}}$ or $\boxed{y = e^{\frac{1}{2}x^2 - 3x + 5/2}}$

(Ic) $(\underbrace{\cos y + y\cos x}_{M})\,dx + (\underbrace{\sin x - x\sin y}_{N})\,dy = 0, \quad y(0) = -1.$

This is exact: $M_y = -\sin y + \cos x$
$N_x = \cos x - \sin y$ Match!

$$\psi = \int M\,dx = \int (\cos y + y\cos x)\,dx$$

$$= x\cos y + y\sin x + h(y)$$

$$\psi_y = N \Rightarrow \quad -x\sin y + \sin x + h'(y) = \sin x - x\sin y$$

$$\Rightarrow \quad h'(y) = 0 \quad \Rightarrow \quad h(y) = c$$

general (implicit) solution:

$$x\cos y + y\sin x = c$$

$$y(0) = 1 \Rightarrow \quad 0\cdot\cos 1 + 1\sin 0 = c \quad \Rightarrow c = 0$$

Final answer $\boxed{x\cos y + y\sin x = 0}$

(Id) $y'' - 2y' - 8y = 0, \quad y(0) = 3, \; y'(0) = \pi.$

This is 2nd order, linear, homogeneous, with constant coefficients: solutions are $y = e^{rx}$ where r satisfies
$r^2 - 2r - 8 = 0.$

$$(r-4)(r+2) = 0 \Rightarrow \quad r = 4, \; r = -2$$

general solution: $y = c_1 e^{4x} + c_2 e^{-2x}.$

$$y(0) = 3 \Rightarrow \quad 3 = c_1 + c_2$$

$$y' = 4c_1 e^{4x} - 2c_2 e^{-2x}$$

$$y'(0) = \pi \Rightarrow \quad \left[\begin{array}{l} \pi = 4c_1 - 2c_2 \\ 6 = 2c_1 + 2c_2 \end{array}\right.$$ ← twice the 1st equation

add equations: $\pi + 6 = 6c_1 \Rightarrow c_1 = \frac{\pi+6}{6} = \frac{\pi}{6} + 1$

$$c_2 = 3 - c_1 = \frac{18 - \pi - 6}{6} = \frac{12-\pi}{6} = 2 - \frac{\pi}{6}$$

$$\boxed{y = \left(\frac{\pi}{6} + 1\right)e^{4x} + \left(2 - \frac{\pi}{6}\right)e^{-2x}}$$

(1e) $4y'' + 4y' + y = 0, \quad y(0) = 1, \; y'(0) = 3$

$4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2 = 0 \quad r = -\frac{1}{2}$

general solution $\quad y = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$

$y(0) = 1 \Rightarrow \quad 1 = c_1 e^0 + c_2 \cdot 0 \cdot e^0$

$\boxed{c_1 = 1}$

Find y': $\quad y' = -\frac{1}{2} e^{-\frac{1}{2}x} + c_2\left[x \cdot \left(-\frac{1}{2}\right) e^{-\frac{1}{2}x} + e^{-\frac{1}{2}x}\right]$

product rule!

$y'(0) = 3$: $\quad 3 = -\frac{1}{2} + c_2\left[0 + e^0\right]$

$3 = -\frac{1}{2} + c_2 \quad \boxed{c_2 = \frac{7}{2}}$

Final answer: $\boxed{y = e^{-\frac{1}{2}x} + \frac{7}{2} x e^{-\frac{1}{2}x}}$

(2) $(x+y)\,dx + (ax+2y)\,dy = 0.$

(a) Find a to be exact: $\quad M_y = 1, \quad N_x = a$

so we want $\boxed{a = 1}$.

(b) $\Psi = \int M\,dx = \int (x+y)\,dx = \frac{1}{2}x^2 + xy + h(y)$

$\Psi_y = N \Rightarrow \quad 0 + x + h'(y) = x + 2y$

$h'(y) = 2y \Rightarrow h(y) = y^2$

general solution: $\boxed{\frac{1}{2}x^2 + xy + y^2 = C}$

(c) $y(0) = 1 \Rightarrow \quad 0 + 0 + 1 = C \quad \Rightarrow C = 1$

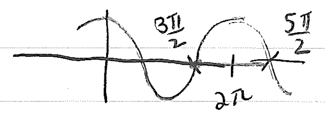
$y^2 + xy + \frac{1}{2}x^2 - 1 = 0$ solve for y, quad. formula:

$$y = \frac{-x \pm \sqrt{x^2 - 4(1)(\frac{1}{2}x^2 - 1)}}{2} = \frac{-x \pm \sqrt{x^2 - 2x^2 + 4}}{2}$$

Final answer: $\boxed{y = \frac{-x + \sqrt{4 - x^2}}{2}}$ ← only positive root satisfies $y(0) = 1$.

(3) (a) $y' - \frac{\sin t}{\cos t} y = 3t, \quad y(2\pi) = 0$

rewritten in correct form. $p(t) = \frac{-\sin t}{\cos t}$ is continuous where $\cos t \neq 0$:



$g(t) = 3t$ continuous everywhere.

The solution is guaranteed to exist and be unique in the interval $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, which includes 2π.

(b) $y' + \frac{5e^{3t}}{t^2 - 81} y = \frac{\sin t}{t^2 - 81}, \quad y(10) = 10\pi$

For continuity, avoid $t = \pm 9$. answer: $\boxed{(9, \infty)}$

(4) $Q(t)$ = pounds of salt at time t.

$\frac{dQ}{dt}$ = rate in − rate out

concentration · rate   concentration · rate

$$= \frac{1}{5}(1 + \cos t) \frac{\text{lbs}}{\text{gal}} \cdot 9 \frac{\text{gal}}{\text{hr}} - \frac{Q(t)}{600 + 3t} \text{lbs}/\text{gal} \cdot 6 \frac{\text{gal}}{\text{hr}}$$

$$\boxed{\frac{dQ}{dt} = \frac{9}{5}(1 + \cos t) - \frac{2}{200 + t} Q(t)}$$

(b) $\frac{dQ}{dt} + \frac{2}{200 + t} Q(t) = \frac{9}{5}(1 + \cos t)$

integrating factor: $\mu = e^{\int \frac{2}{200+t} dt} = e^{2 \ln|200+t|} = \boxed{(200+t)^2}$

(c)

$$\underbrace{(200+t)^2 \frac{dQ}{dt} + 2(200+t) Q(t)} = \frac{9}{5}(1 + \cos t)(200+t)^2$$

$$\frac{d}{dt}\left[Q(200+t)^2\right] = \frac{9}{5}(1 + \cos t)(40000 + 400t + t^2)$$

$$Q(200+t)^2 = \int \frac{9}{5}\left(40000 + 400t + t^2 + 40000 \cos t + 400t \cos t + t^2 \cos t\right) dt$$

To finish, integrate all terms. Last two need integration by parts.

4c continued... with patience, the general solution is

$$\boxed{Q(t) = \frac{9}{5}\left(\frac{1}{3}(200+t) + \sin t + \frac{2\cos t}{200+t} - \frac{2\sin t}{(200+t)^2}\right) + \frac{c}{(200+t)^2}}$$

Applying the initial condition, $Q(0) = 5$, yields $\boxed{C = 4600720}$

The tank will overflow when total volume is 1500 gallons, a gain of 900 gallons, which takes 300 hours. We want $Q(300)$, which you can compute to be $\boxed{279.797 \text{ pounds}}$

(5) $y'' + 4y = 0$

(a) $y_1 = \cos 2t$, $y_1' = -2\sin 2t$, $y_1'' = -4\cos 2t$
check: $-4\cos 2t + 4(\cos 2t) = 0$ yes!
$y_2 = \sin 2t$, $y_2' = 2\cos 2t$, $y_2'' = -4\sin 2t$
check: $-4\sin 2t + 4(\sin 2t) = 0$, also true.

For independence check the Wronskian $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2(\cos 2t)^2 + 2(\sin 2t)^2$$

$$= 2(\cos^2 2t + \sin^2 2t) = 2(1) = 2. \quad \text{not} \equiv 0$$

(b) The original equation is 2^{nd} order, linear, and homogeneous. Since y_1, y_2 are linearly independent, $y = c_1 y_1 + c_2 y_2$ is the general solution.

(6) $yy'' + (y')^2 = 0.$

(a) $y_1 = 1, \quad y_1' = 0, \quad y_1'' = 0$

check: $1 \cdot 0 + 0^2 = 0$ yes!

$y_2 = t^{\frac{1}{2}} \quad y_2' = \frac{1}{2} t^{-\frac{1}{2}} \quad y_2'' = -\frac{1}{4} t^{-3/2}$

check: $(t^{\frac{1}{2}})(-\frac{1}{4} t^{-3/2}) + (\frac{1}{2} t^{-\frac{1}{2}})^2$

$= -\frac{1}{4} t^{-1} + \frac{1}{4} t^{-1} = 0.$ yes!

$$W(y_1, y_2) = \begin{vmatrix} 1 & t^{\frac{1}{2}} \\ 0 & \frac{1}{2} t^{-\frac{1}{2}} \end{vmatrix} = \frac{1}{2} t^{-\frac{1}{2}}$$

<u>not</u> identically zero,

so they <u>are</u> linearly independent.

(b) $y = 1 + t^{\frac{1}{2}} \quad y' = \frac{1}{2} t^{-\frac{1}{2}} \quad y'' = -\frac{1}{4} t^{-3/2}$

$(1 + t^{\frac{1}{2}})(-\frac{1}{4} t^{-3/2}) + (\frac{1}{2} t^{-\frac{1}{2}})^2$

$= -\frac{1}{4} t^{-3/2} \underbrace{- \frac{1}{4} t^{-1} + \frac{1}{4} t^{-1}} = -\frac{1}{4} t^{-3/2} \neq 0.$

Thus $y = 1 + t^{\frac{1}{2}}$ is NOT a solution.

The original equation is homogeneous, but is <u>NOT</u> linear: we do <u>not</u> expect the general solution to be of the form $y = c_1 y_1 + c_2 y_2$.

Whew! We get a sticker. :)