SOLUTIONS

MA 214—04 Exact Equations

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if $M_y = N_x$

1. One of these equations is exact and the other is not. Test them both, and then solve the exact equation. Leave your answer in implicit form.
   a. $(e^x \sin y + 3y)\,dx + (3x + e^x \sin y)\,dy = 0$
   b. $(2xy^2 + 2y) + (2x^2y + 2x + y)\dfrac{dy}{dx} = 0$

(a) $M(x,y) = e^x \sin y + 3y$ $\quad N(x,y) = 3x + e^x \sin y$ $\quad M_y \neq N_x$, so the equation is not exact!

$M_y = e^x \cos y + 3$ $\quad N_x = 3 + e^x \sin y$

(b) $(2xy^2 + 2y)\,dx + (2x^2y + 2x + y)\,dy = 0$

$M(x,y) = 2xy^2 + 2y$ $\quad N(x,y) = 2x^2y + 2x + y$

$M_y = 4xy + 2$ $\quad N_x = 4xy + 2$ $\quad M_y = N_x$, so this is an exact equation!

$\psi = \int M\,dx = \int (2xy^2 + 2y)\,dx = x^2y^2 + 2xy + h(y)$

$2x^2y + 2x + y = N = \psi_y = 2x^2y + 2x + h'(y)$, so $h'(y) = y$.

Therefore, $h(y) = \int y\,dy = \frac{1}{2}y^2$.

So, solutions are given by

$\psi(x,y) = x^2y^2 + 2xy + \frac{y^2}{2} = c$

2. Consider the equation $x^2y^3dx + x(1+y^2)\,dy = 0$.
   a. Show the equation is *not* exact.
   b. Multiply the equation by the integrating factor $\mu(x,y) = \dfrac{1}{xy^3}$, and show the new equation *is* exact. (If you have time, solve the new equation.)

(a) $M(x,y) = x^2y^3$ $\quad N(x,y) = x(1+y^2)$ $\quad M_y \neq N_x$, so the equation is not exact

$M_y = 3x^2y^2$ $\quad N_x = 1 + y^2$

(b) Multiplying by $\mu(x,y) = \frac{1}{xy^3}$, we have:

$x\,dx + \dfrac{1+y^2}{y^3}\,dy = 0$

Now, $\tilde{M}(x,y) = x$ $\quad \tilde{N}(x,y) = \frac{1}{y^3} + \frac{1}{y}$

$\tilde{M}_y = 0$ $\quad \tilde{N}_x = 0$

$\tilde{M}_y = \tilde{N}_x = 0$, so the new equation is exact

Over!

$\psi = \int \tilde{N}\, dy = \int (y^{-3} + y^{-1})\, dy = \frac{y^{-2}}{-2} + \ln|y| + f(x)$

$x = \tilde{m} = \frac{\partial \psi}{\partial x} = f'(x)$

So, $f(x) = \int x\, dx = \frac{1}{2}x^2$

Therefore, $\psi = \frac{1}{2}x^2 - \frac{1}{2y^2} + \ln|y|$

So, we have the solutions

$\psi(x,y) = \frac{x^2}{2} - \frac{1}{2y^2} + \ln|y| = c$

and $y = 0$ (since we divided by y in our integrating factor)