SOLUTIONS

MA 214—04 2nd order homogeneous with real roots & the Wronskian

1. Write the general solution for each equation.
   a. $y'' + 14y' + 49y = 0$
   b. $2y'' - 5y' - 3y = 0$

For both, we find the characteristic equation and then use the appropriate form for the general solution.

(a) $r^2 + 14r + 49 = 0$
$(r+7)^2 = 0$, repeated roots of $r = -7$

$$\boxed{y(t) = C_1 e^{-7t} + C_2 t e^{-7t}}$$

(b) $2r^2 - 5r - 3 = 0$
$(2r+1)(r-3) = 0$
So, we have roots $r = -1/2, 3$.

$$\boxed{y(t) = C_1 e^{-1/2\, t} + C_2 e^{3t}}$$

2. Find and simplify the Wronskian of the pair of functions:
   a. $y_1 = \sin x$ and $y_2 = \cos x$
   b. $y_1 = e^x \sin x$ and $y_2 = e^x \cos x$

(a) $W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -(\sin^2 x + \cos^2 x) = \boxed{-1}$

(b) $W(y_1, y_2) = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & e^x \cos x - e^x \sin x \end{vmatrix}$

$= e^{2x} \cancel{\sin x \cos x} - e^{2x} \sin^2 x - e^{2x} \cancel{\sin x \cos x} - e^{2x} \cos^2 x$

$= -e^{2x}(\sin^2 x + \cos^2 x) = \boxed{-e^{2x}}$

3. Consider the equation $t^2 y'' - 2y = 0$ for $t > 0$.
   a. Verify ~~$y_1 = 1$ and $y_2 = t^{1/2}$~~ $y_1 = t^2$ and $y_2 = t^{-1}$ are solutions.
   b. Explain why we know $y = C_1 t^2 + C_2 t^{-1}$ is the general solution.

(a) $y_1 = t^2$
$y_1' = 2t$
$y_1'' = 2$
Thus $t^2 y_1'' - 2y_1$
$= t^2(2) - 2t^2 = 0$

$y_2 = t^{-1}$
$y_2 = -t^{-2}$
$y_3 = 2t^{-3}$
$t^2 y_2'' - 2y_2 = t^2(2t^{-3}) - 2(t^{-1}) = 2t^{-1} - 2t^{-1} = 0$

Therefore, y_1 and y_2 are solutions to the differential equation.

(b) The equation is equal to 0 and thus homogenous; it is also linear and 2nd order.
Moreover, $W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = t^2(-t^{-2}) - 2t(t^{-1}) = -1 - 2 = -3 \neq 0$
So, y_1 and y_2 are linearly independent, so we have met the conditions for the general solution to be as shown.