General First Order Linear Equations

Example 1: Find the general solution to the differential equation

$$(1+t^2)y' + 4ty = \frac{1}{(1+t^2)^2}$$

First we need to put it in the correct form by dividing both sides of the equation by $1+t^2$ which gives

$$y' + \frac{4t}{1+t^2}y = \frac{1}{(1+t^2)^3}$$

The integrating factor is $m(t) = e^{\int \frac{4t}{1+t^2}\,dt}$. After integration by substitution we get $m(t) = (1+t^2)^2$. Therefore, we have

$$\begin{aligned}
y(t) &= \frac{1}{(1+t^2)^2}\int (1+t^2)^2 \frac{1}{(1+t^2)^3}\,dt \\
&= \frac{1}{(1+t^2)^2}\int \frac{1}{1+t^2}\,dt \\
&= \frac{1}{(1+t^2)^2}(\tan^{-1}(t) + C) \\
&= \frac{1}{(1+t^2)^2}\tan^{-1}(t) + C\frac{1}{(1+t^2)^2}
\end{aligned}$$

Note: $\lim_{t\to\infty} y(t) = 0$.

Newton's Law of Cooling (Sec 1.3.3)

Newton's law of cooling says that the rate at which an object cools to an ambient temperature, A, is proportional to the difference between the objects temperature at time t, $T(t)$ and the ambient temperature. The differential equation modeling this situation is

$$\frac{dT}{dt} = k(A - T)$$

where k is the proportionality constant determined by the initial conditions of the problem.

Example 1: A thermometer is taken from a room of constant temperature $21°C$ and brought outside where the temperature is $-6°C$. After 1 minute the thermometer reads $7°C$.

a. **What will the reading be after 1 more minute?**

The information in the problem is equivalent to solving the initial value problem $\frac{dT}{dt} = k(-6-T)$, with $T(0) = 21$ and $T(1) = 7$. We know the general solution to the differential equation is $T(t) = Ce^{-kt} - 6$ since the solution to $\frac{dT}{dt} = aT + b$ is $T(t) = Ce^{at} - \frac{b}{a}$. The condition $T(0) = 21$ gives

$$21 = Ce^0 - 6 \implies 27 = C$$

The condition $T(1) = 7$ lets us solve for k.

$$7 = 27e^{-k(1)} - 6 \implies k = -\ln\left(\frac{13}{27}\right)$$

. Now that we have found C and k, we know that

$$T(t) = 27e^{\ln\left(\frac{13}{27}\right)t} - 6 = 27\left(\frac{13}{27}\right)^t - 6$$

Therefore,

$$T(2) = 27\left(\frac{13}{27}\right)^2 - 6 \approx -2.986282579°C$$

b. **When will the thermometer read $-5°C$?**

We want to solve

$$\begin{aligned} -5 &= 27\left(\frac{13}{27}\right)^t - 6 \\ \implies 1 &= 27\left(\frac{13}{27}\right)^t \\ \implies t &= \frac{\ln(\frac{1}{27})}{\ln(\frac{13}{27})} \approx 4.5093 \end{aligned}$$

Rate In/ Rate Out: (1.4.2)

Example 2: A tank contains 70kg of salt and 1000L of water. A solution of concentration 0.035 kg of salt per liter enters the tank at a rate of 8 liters per minute. The solution is mixed and drained at the same rate.

a. What is the initial concentration of the solution in the tank?

Concentration $= \frac{70\,kg}{1000\,L} = .07\,\frac{kg}{L}$.

b. Find the amount of salt in the tank after 4 hours.

To do this, we need to model the situation with a differential equation and then solve the equation to find a function that outputs the amount of salt in the tank at time t.

Let $Q(t)$ be the quantity of salt at time t. In this problem, the rate of change of salt in the tank is equal to the rate at which the salt is entering the tank minus the rate at which the salt is leaving the tank.

$$\frac{dQ}{dt} = \text{rate in - rate out}$$

The rate in is the product of the concentration of the input solution and the flow rate of the input solution.

$$\begin{aligned} \text{rate in} &= \text{concentration} \cdot \text{flow rate} \\ &= .035\frac{kg}{L} \cdot 8\frac{L}{min} \\ &= .28\frac{kg}{min} \end{aligned}$$

The rate out is the product of the concentration of the output solution and the flow rate of the output solution.

$$\begin{aligned}\text{rate out} &= \text{concentration} \cdot \text{flow rate}\\ &= \frac{Q}{1000}\frac{kg}{L} \cdot 8\frac{L}{min}\\ &= .008Q\frac{kg}{min}\end{aligned}$$

Therefore, we have the differential equation

$$\frac{dQ}{dt} = .28 - .008Q$$

with initial condition $Q(0) = 70$.
The general solution to this differential equation is
$Q(t) = Ce^{-.008t} + \frac{.28}{.008} = Ce^{-.008t} + 35$. Using the initial condition, we get
$70 = Ce^0 + 35 \implies C = 35$. Therefore,

$$Q(t) = 35e^{-.008t} + 35$$

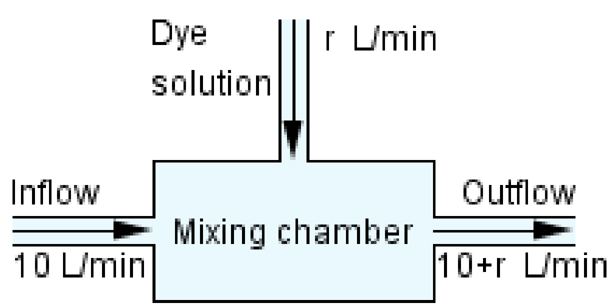
To finish the problem, first convert 4 hours to 240 minutes. Then we have

$$Q(240) = 35e^{-.008*240} + 35 \approx 40.131244\, kg$$

c. What does the concentration approach as time approaches infinity?

$$\text{Concentration} = \frac{\text{Quantity}}{\text{Volume}} = \frac{Q(t)}{1000} = \frac{35e^{-.008t} + 35}{1000} \to \frac{35}{1000}\frac{kg}{L}\ \text{as}\, t \to \infty$$

Example 3: A mixing chamber initially contains 5 L of a clear liquid. Clear liquid flows into the chamber at a rate of 10 L per min. A dye solution having a concentration of 0.75 kg per L is injected into the mixing chamber at a constant rate of r L per min. When the mixing process is started, the well-stirred mixture is pumped from the chamber at a rate of $10 + r$ L per min.



a. Develop a mathematical model for the mixing process

Let $Q(t)$ be represent the amount of dye in kg in the mixture at time t.

$$\frac{dQ}{dt} = \text{rate in - rate out}$$

$$\begin{aligned}
\text{rate in} &= \text{concentration left } \cdot \text{flow rate left} + \text{concentration top } \cdot \text{flow rate top} \\
&= 0\frac{kg}{L} \cdot 10\frac{L}{min} + 0.75\frac{kg}{L} \cdot r\frac{L}{min} \\
&= 0.75r\frac{kg}{min}
\end{aligned}$$

$$\begin{aligned}
\text{rate out} &= \text{concentration out } \cdot \text{flow rate out} \\
&= \frac{Q}{5}\frac{kg}{L} \cdot (10+r)\frac{L}{min} \\
&= \frac{Q(10+r)}{5}\frac{kg}{min}
\end{aligned}$$

Therefore, we have the model

$$\frac{dQ}{dt} = 0.75r - \frac{(10+r)}{5}Q$$

b. The objective is to obtain a dye concentration in the outflow mixture of 0.6 kg per L. What injection rate r is required to achieve this equilibrium solution?

There is always 5 L of liquid in the tank, so the concentration of dye is $\frac{Q}{5}\frac{kg}{L}$ so we want $\frac{Q}{5} = 0.6$. In this situation, the rate of change of the dye in the mixture is constant. So we have,

$$0 = 0.75r - .6(10+r) \implies r = \frac{6}{.15} = 40$$

Note: This equilibrium value of r would not be different if the fluid in the chamber at $t = 0$ contained some dye.

c. How many minutes will it take for the outflow concentration to rise to within 2% of the desired concentration of 0.6 kg per L?

Because the solution is clear to start with, being within 2% of the desired concentration means we have at least 98% of the desired concentration. The first thing we should do is solve for $Q(t)$. The general solution to the differential equation $\frac{dQ}{dt} = 30 - 10Q$ is $Q(t) = Ce^{-10t} + 3$. The initial condition is $Q(0) = 0$ which means $C = -3$. We want the concentration $\frac{Q}{5}$ to be at least 98% of 0.6. In other words, we want to solve

$$\begin{aligned}
\frac{-3e^{-10t} + 3}{5} &= (0.98)(0.6) \\
\implies e^{-10t} &= \frac{(0.98)(0.6)(5) - 3}{-3} \\
\implies t &= \frac{\ln\left(\frac{(0.98)(0.6)(5)-3}{-3}\right)}{-10} \approx 0.391202
\end{aligned}$$