Variation of Parameters Worksheet, #3 Solutions

$t^2 y'' - 2y = 3t^2 - 1$ for $t > 0$

First, we put this in the form $y'' + q(t)y' + r(t)y = g(t)$.

$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2} \quad (*)$$

So, the associated homogeneous equation is $y'' - \frac{2}{t^2} y = 0$.

We check $y_1 = t^2$ and $y_2 = t^{-1}$ solve this:

$y_1' = 2t$

$y_1'' = 2$

$y_1'' - \frac{2}{t^2} y' = 2 - \frac{2}{t^2}(t^2) = 2 - 2 = 0$ ✓

$y_2' = -t^{-2}$

$y_2'' = 2t^{-3}$

$y_2'' - \frac{2}{t^2} y_2 = 2t^{-3} - \frac{2}{t^2}\left(\frac{1}{t}\right)$

$= \frac{2}{t^3} - \frac{2}{t^3} = 0$ ✓

From variation of parameters, a particular solution is given by:

$$Y_p(t) = -y_1(t) \int \frac{y_2(t) g(t)}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t) g(t)}{W(y_1, y_2)} dt \quad (**)$$

From $(*)$, $g(t) = 3 - \frac{1}{t^2}$.

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = (t^2)(-t^{-2}) - (2t)(t^{-1}) = -1 - 2 = -3$$

So, proceeding from $(**)$:

$$Y_p(t) = -t^2 \int \frac{(t^{-1})(3 - t^{-2})}{-3} dt + t^{-1} \int \frac{t^2(3 - t^{-2})}{-3} dt$$

$$= \frac{t^2}{3} \int 3t^{-1} - t^{-3} dt - \frac{t^{-1}}{3} \int 3t^2 - 1 \, dt$$

$$= \frac{t^2}{3}\left[3\ln(t) + \frac{t^{-2}}{2}\right] - \frac{t^{-1}}{3}\left[t^3 - t\right] = t^2 \ln(t) + \frac{1}{6} \underbrace{- \frac{t^2}{3}}_{\substack{\text{this is a constant} \\ \text{times } y_1(t), \\ \text{so it solves} \\ \text{homogeneous eqn.}}} + \frac{1}{3}$$

So, $Y_p(t) = t^2 \ln(t) + \frac{1}{2}$.

Therefore, the general solution to $(*)$ is:

$$\boxed{Y(t) = c_1 t^2 + c_2 t^{-1} + t^2 \ln(t) + \frac{1}{2}}$$

Variation of Parameters Worksheet, #4 Solutions

$$y'' - 4y' + 4y = t^2 e^{2t} \quad (*)$$

So, $g(t) = t^2 e^{2t}$ and the associated homogeneous eqn. is $y'' - 4y' + 4y = 0$.

Solving homogeneous eqn.:

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

So, $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$ are linearly independent solutions to the homogeneous eqn.

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix} = e^{4t} + 2te^{4t} - 2te^{4t} = e^{4t}$$

Using variation of parameters:

$$Y_p(t) = -y_1(t) \int \frac{y_2(t) g(t)}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t) g(t)}{W(y_1, y_2)} dt$$

$$= -e^{2t} \int \frac{(te^{2t})(t^2 e^{2t})}{e^{4t}} dt + te^{2t} \int \frac{(e^{2t})(t^2 e^{2t})}{e^{4t}} dt$$

$$= -e^{2t} \int \frac{t^3 e^{4t}}{e^{4t}} dt + te^{2t} \int \frac{t^2 e^{4t}}{e^{4t}} dt$$

$$= -e^{2t} \int t^3 \, dt + te^{2t} \int t^2 \, dt = -e^{2t} \left(\frac{t^4}{4}\right) + te^{2t}\left(\frac{t^3}{3}\right)$$

$$= \frac{-t^4 e^{2t}}{4} + \frac{t^4 e^{2t}}{3} = \frac{-3t^4 e^{2t} + 4t^4 e^{2t}}{12} = \frac{1}{12} t^4 e^{2t}$$

Therefore, the general solution to the differential equation $(*)$ is:

$$\boxed{Y(t) = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{12} t^4 e^{2t}}$$