ANSWER KEY

These problems should help prepare for the second exam. This collection is NOT comprehensive! Look over the previous worksheets, homework, class examples and the second quiz.

1. Calculate the Laplace transform of the following functions using the definition.

   a. $f(t) = 3t + 2$

   $\mathcal{L}\{3t+2\} = \frac{2}{s} + \frac{3}{s^2}$

   b. $f(t) = e^{10t}$

   $\mathcal{L}\{e^{10t}\} = \frac{1}{s-10}$

2. Find the inverse Laplace transform of the following functions using the table.

   a. $F(s) = \dfrac{2}{s^2 + 3s - 4}$

   $f(t) = \frac{2}{5} e^{t} - \frac{2}{5} e^{-4t}$

   b. $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2 + 4)}$

   $f(t) = 3 - 2\sin(2t) + 5\cos(2t)$

3. Solve the initial value problems using Laplace transforms.

   a. $y'' - y' - 6y = 0;\ y(0) = 1,\ y'(0) = -1$

   $y(t) = \frac{1}{5} e^{3t} + \frac{4}{5} e^{-2t}$

   b. $y'' - 4y' + 4y = 0;\ y(0) = 1,\ y'(0) = 1$

   $y(t) = e^{2t} - te^{2t}$

   c. $y'' - 2y' + 2y = \cos t;\ y(0) = 1,\ y'(0) = 0$

   $y(t) = \frac{1}{5}(\cos t - 2\sin t + 4e^{t}\cos t - 2e^{t}\sin t)$

4. A mass of 100g stretches a spring 5cm. If the mass is set in motion from equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position u of the mass at any time t.

$$y = \frac{5}{7}\sin(14t) \text{ cm}$$

5. Solve the initial value problem $y'' + 4y' + 5y = 0$; $y(0) = 1$, $y'(0) = 0$.

$$y(t) = e^{-2t}\cos t + 2e^{-2t}\sin t$$

6. Use the method of undetermined coefficients to find the general solution of $y'' + 2y' + 5y = 3\sin(2t)$.

$$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)$$

7. Use the method of undetermined coefficients to solve the initial value problem $y'' - 2y' + y = te^t + 4$; $y(0) = 1$, $y'(0) = 1$.

$$y(t) = 4te^t - 3e^t + \frac{1}{6}t^3e^t + 4$$

8. Use the method of variation of parameters to find a general solution to the equation $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$.

$$y(t) = c_1 e^{-2t} + c_2 te^{-2t} - e^{-2t}\ln(t)$$

Chloe Darnell

1. (a) $f(t) = 3t + 2$

$$\mathcal{L}\{3t+2\} = \int_0^\infty (3t+2)e^{-st}\,dt$$

$$= \lim_{R\to\infty} \int_0^R (3t+2)e^{-st}\,dt$$

$u = 3t+2$ $\quad dv = e^{-st}dt$ $\quad uv - \int v\,du$

$du = 3dt$ $\quad v = \frac{e^{-st}}{-s}$

$$\lim_{R\to\infty}\left(\frac{(3t+2)e^{-st}}{-s}\Big]_0^R + \int_0^R \frac{3e^{-st}}{s}\,dt\right)$$

$$\lim_{R\to\infty} \cancel{\frac{(3R+2)e^{-sR}}{-s}}^{\,0} - \frac{(0+2)e^0}{-s} + \int_0^R \frac{3e^{-st}}{s}\,dt$$

$$\frac{2}{s} + \lim_{R\to\infty} \frac{3}{s}\int_0^R e^{-st}\,dt$$

$$\frac{2}{s} + \lim_{R\to\infty} \frac{3}{s}\cdot\frac{e^{-st}}{-s}\Big]_0^R$$

$$\frac{2}{s} + \lim_{R\to\infty} \cancel{\frac{3e^{-sR}}{-s^2}}^{\,0} + \frac{3e^0}{s^2}$$

$$\frac{2}{s} + \frac{3}{s^2}, \quad s > 0$$

1B $f(t) = e^{10t}$.

$$F(s) = \int_0^\infty e^{-st} e^{10t} \, dt$$

$$= \int_0^\infty e^{(10t - st)} \, dt$$

$$= \lim_{R\to\infty} \int_0^R e^{(10-s)t} \, dt$$

$$= \lim_{R\to\infty} \frac{1}{10-s} e^{(10-s)t} \Big|_0^R$$

$$= \lim_{R\to\infty} \left[\underbrace{\frac{1}{10-s} e^{(10-s)R}}_{\text{if } 10-s<0,^* \text{ this } \to 0 \text{ as } R\to\infty} - \frac{1}{10-s} \underbrace{e^{(10-s)\cdot 0}}_{e^0 = 1} \right]$$

$$= 0 - \frac{1}{10-s}(1) = \frac{-1}{10-s}$$

$$= \boxed{\frac{1}{s-10}}$$

* works for $s > 10$

2.) A. Find inverse laplace transform of the following functions using the ~~definition~~ table

$F(s) = \frac{2}{s^2+3s-4} \Rightarrow \frac{A}{(s+4)} + \frac{B}{(s-1)} = \frac{2}{(s+4)(s-1)}$

s^2+3s-4

$(s+4)(s-1)$

$s=-4$

$s=1$

$A + \frac{B(s+4)}{(s-1)} = \frac{2}{s-1}$ plug in $s=4$

$A + \cancel{\frac{B(-4+4)}{s-1}}^{0} = \frac{2}{-5}$

$A = -2/5$

$\frac{A(s-1)}{s+4} + B = \frac{2}{s+4}$

plug in $s=1$

$B = \frac{2}{5}$

$\frac{-2}{5}\left(\frac{1}{s+4}\right) + \frac{2}{5}\left(\frac{1}{s-1}\right)$

$* \; y = \frac{-2}{5}e^{-4t} + \frac{2}{5}e^{t} \; *$

(2) b) $F(s) = \dfrac{8s^2-4s+12}{s(s^2+4)}$

$$\frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{8s^2-4s+12}{s(s^2+4)}$$

$$A(s^2+4) + (Bs+C)s = 8s^2-4s+12$$

$\underline{s=2i}$

$$(B\cancel{s}+C)2i = 8(-4) - 8i + 12$$

$$-4B + 2Ci = -32 - 8i + 12$$

$$-4B + 2Ci = -20 - 8i$$

$$C = -4$$

$$-4B = -20$$

$$B = 5$$

$\underline{s=0}$

$$4A = 12$$

$$A = 3$$

$$\frac{3}{s} + \frac{5s-4}{s^2+4}$$

$$= \frac{3}{s} + \frac{5s}{s^2+4} - 2\frac{2}{s^2+4}$$

$$= 3 + 5\cos(2t) - 2\sin(2t)$$

2b) $F(s) = \dfrac{8s^2 - 4s + 12}{s(s^2+4)} \Rightarrow \dfrac{A}{s} + \dfrac{Bs+C}{s^2+4}$

$A(s^2+4) + Bs^2 + Cs = 8s^2 - 4s + 12$

$\underline{s=0};\ 4A + 0 + 0 = 0 - 0 + 12$

$\underline{A = 3}$

$\underline{s = 2i};\ 3((-4)+4) + B(-2) + C(2i) = -32 - 8i + 12$

$\underline{B = 5} \quad \underline{C = -4}$

$\dfrac{3}{s} + \dfrac{5s - 4}{s^2 + 2^2} = \dfrac{3}{s} + \dfrac{5s}{s^2+2^2} - \dfrac{4}{s^2+2^2}$

$f(t) = 3 + 5\cos(2t) - 2\sin(2t)$

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2b) $\dfrac{8s^2-4s+12}{s(s^2+4)}$; $\dfrac{8s^{\cancel{2}}}{\cancel{s}(s^2+2^2)} - \dfrac{4\cancel{s}}{\cancel{s}(s^2+2^2)} + \dfrac{12}{s(s^2+2^2)}$

$$8\left(\frac{s}{s^2+2^2}\right) - 4\left(\frac{1}{s^2+2^2}\right) + \frac{12}{s(s^2+2^2}$$

$$\frac{4}{2}\left(\frac{2}{s^2+2^2}\right) + \frac{A}{s} + \frac{Bs+C}{s^2+4} \quad *$$

$A=3$, $B=-3$, $C=0$

$$\frac{3}{s} + \frac{-3s}{s^2+2^2}$$

$$3\left(\frac{1}{s}\right) + -3\left(\frac{s}{s^2+2^2}\right)$$

$$8(\cos 2t) - 2(\sin 2t) + 3 - 3(\cos 2t)$$

$$\boxed{f(t) = 5\cos(2t) - 2\sin(2t) + 3}$$

* Solving Partial *

$A(s^2+4) + (Bs+C)(s) = 12$

$As^2 + 4A + Bs^2 + Cs = 12$

$As^2 + Bs^2 + Cs + 4A = 12$

$s^2(A+B) + Cs + 4A = 12$

$(A+B) = 0$

$C = 0$

$4A = 12 \Rightarrow \underline{A=3}$

$\underline{B=-3}$

Problem 3-a

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$s^2(Y(s)) - s(y(0)) - y'(0) - s(Y(s)) + y(0) - 6(Y(s)) = 0$

$s^2(Y(s)) - s + 1 - sY(s) + 1 - 6Y(s) = 0$

$Y(s)(s^2 - s - 6) = s - 2 \qquad Y(s) = \dfrac{s-2}{(s-3)(s+2)}$

$\dfrac{s-2}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2}$

$s - 2 = A(s+2) + B(s-3) \quad @ s = -2$

$-4 = 0 + B(-5) \qquad B = 4/5$

$@ s = 3$

$1 = A(5) + 0 \qquad A = 1/5$

$Y(s) = \frac{1}{5}\left(\frac{1}{s-3}\right) + \frac{4}{5}\left(\frac{1}{s+2}\right)$

$f(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$

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3.

b) $y'' - 4y' + 4y = 0$; $y(0) = 1$, $y'(0) = 1$

$$s^2 Y(s) - s - 1 - 4sY(s) + 4 + 4Y(s) = 0$$

$$s^2 Y(s) - 4sY(s) + 4Y(s) = s - 3$$

$$Y(s) = \frac{s-3}{s^2 - 4s + 4}$$

$$Y(s) = \frac{s-3}{(s-2)^2}$$

$$Y(s) = \frac{(s-2) - 1}{(s-2)^2}$$

$$Y(s) = \frac{(s-2)}{(s-2)^2} - \frac{1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

$$Y(s) = \frac{1}{(s-2)} - \frac{1}{(s-2)^2}$$

$$y(t) = e^{2t} - te^{2t}$$

table #4 table #5

4. $0.1y'' + 19.6y = 0$ $\qquad y(0)=0, \quad y'(0)=10$ cm/s

$$y'' + 196y = 0$$

$$r^2 + 196 = 0$$

$$r = \pm 14i$$

$$y = c_1 \cos(14t) + c_2 \sin(14t)$$

$$0 = c_1$$

$$y' = -14c_1 \sin(14t) + 14c_2 \cos(14t)$$

$$10 = 14c_2$$

$$c_2 = \frac{5}{7}$$

$$\boxed{y = \frac{5}{7}\sin(14t)} \text{ cm}$$

Setting up equation:

$100g = .1kg = m$

no damping, so $\gamma = 0$ $\qquad$ convert g to cm/s^2

$$k = \frac{mg}{L} = \frac{.1(980)}{5} = 19.6$$

$$my'' + \gamma y' + ky = 0 \Rightarrow .1y'' + 19.6y = 0$$

Starts from equilibrium: $y(0) = 0$

initial velocity 10 cm/s downward: $y'(0) = 10$

5. $y'' + 4y' + 5y = 0$

$r^2 + 4r + 5 = 0$

$\dfrac{-4 \pm \sqrt{16 - 4(1)(5)}}{2}$

$\dfrac{-4 \pm \sqrt{-4}}{2}$

$-2 \pm i$

$y = c_1 e^{-2t} \sin t + c_2 e^{-2t} \cos t$

$y(0) = 1$

$1 = c_1 e^0 \sin(0) + c_2 e^0 \cos(0)$

$1 = c_2$

$y = e^{-2t}(c_1 \sin t + c_2 \cos t)$

$y' = -2e^{-2t}(c_1 \sin t + c_2 \cos t) + e^{-2t}(c_1 \cos t - c_2 \sin t)$

$y'(0) = 0$

$0 = -2e^0(c_1 \sin(0) + (1)\cos(0)) + e^0(c_1 \cos(0) - (1)\sin(0))$

$0 = -2(c_1 - 0) + 1(c_1(1) - 0)$

$0 = -2(1) + c_1$

$0 = -2 + c_1$

$c_1 = 2$

$y = 2e^{-2t} \sin t + e^{-2t} \cos t$

8. $y'' + 4y' + 4y = t^{-2} e^{-2t}, \quad t > 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

so... $y_1 = e^{-2t} \quad y_2 = te^{-2t}$

$y = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & t(-2te^{-2t}) + e^{-2t} \end{vmatrix}$$

$= e^{-2t} t(-2e^{-2t}) + e^{-4t} - (-2e^{-2t} te^{-2t})$

$= -2e^{-4t} + e^{-4t} + 2e^{-4t} t$

$W = e^{-4t}$

$$u_1 = -\int \frac{f y_2}{W} = -\int \frac{t^{-2} e^{-2t} t e^{-2t}}{e^{-4t}} = -\int \frac{1}{t} = [-\ln|t|]$$

$$u_2 = \int \frac{f y_1}{W} = \int \frac{t^{-2} e^{-2t} e^{-2t}}{e^{-4t}} = \int \frac{1}{t^2} = \left[-\frac{1}{t}\right]$$

$y_p = u_1 y_1 + u_2 y_2$

$y_p = -(\ln|t|)e^{-2t} \underbrace{- e^{-2t}}$ ← particular solution

matches homogeneous, so absorbed

general solution:

$$\boxed{y = c_1 e^{-2t} + c_2 t e^{-2t} - e^{2t} \ln|t|}$$

Table 1 Laplace Transforms

$f(t)$	$F(s)$
1. 1	$\frac{1}{s}$
2. t	$\frac{1}{s^2}$
3. t^n	$\frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, \ldots$
4. e^{at}	$\frac{1}{s-a}$
5. te^{at}	$\frac{1}{(s-a)^2}$
6. $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad n = 0, 1, 2, \ldots$
7. $\sin bt$	$\frac{b}{s^2+b^2}$
8. $\cos bt$	$\frac{s}{s^2+b^2}$
9. $t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
10. $t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
11. $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
12. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
13. $e^{at} - e^{bt}$	$\frac{a-b}{(s-a)(s-b)}$
14. $ae^{at} - be^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$
15. $(b-c)e^{at} + (c-a)e^{bt} + (a-b)e^{ct}$	$\frac{(a-b)(b-c)(a-c)}{(s-a)(s-b)(s-c)}$
16. $\sin bt - bt \cos bt$	$\frac{2b^3}{(s^2+b^2)^2}$