The Laplace Transform(Sec 6.1)

We know how to solve linear, constant-coefficient ODE's when the inhomogeneous term is continuous by using the methods of undetermined coefficients and variation of parameters. But what if the inhomogeneous term is discontinuous? To deal with the discontinuous case we could adapt the methods we already know. However, the **Laplace transform** technique provides a simpler and more elegant solution.

The main idea is to transform the differential equation for $y(t)$ into an algebraic equation for the Laplace-transformed function $Y(s)$. Then we will solve the algebraic eqaution for $Y(s)$ and use the inverse Laplace transform to find the solution $y(t)$.

The Laplace transform of $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, is defined by the integral transform

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

It is important to notice that the input of the Laplace transform is a function of t (usually representing time) and the output is a function of s (usually representing frequency).

Before we start using the Laplace transform to solve differential equations, we need to practice evaulating the transform of common functions.

Example 1: $\mathcal{L}\{1\} = \frac{1}{s}$, $s > 0$

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}\, dt \\
&= \lim_{R\to\infty} \int_0^R e^{-st}\, dt \\
&= \lim_{R\to\infty} -\frac{e^{-st}}{s}\bigg|_0^R \\
&= \lim_{R\to\infty} -\frac{e^{-sR}}{s} + \frac{1}{s} \\
&= \frac{1}{s}
\end{aligned}$$

Example 2: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at}\, dt \\
&= \lim_{R\to\infty} \int_0^R e^{(a-s)t}\, dt \\
&= \lim_{R\to\infty} \frac{e^{(a-s)t}}{a-s}\bigg|_0^R \\
&= \lim_{R\to\infty} \frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s} \\
&= \frac{1}{s-a}
\end{aligned}$$

Example 3: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$, $s > a$ where $F(s) = \mathcal{L}\{f(t)\}$.

$$\begin{aligned}
\mathcal{L}\{f(t)e^{at}\} &= \int_0^\infty e^{-st}f(t)e^{at}\,dt \\
&= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\
&= F(s-a)
\end{aligned}$$

Example 4: $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$, $s > 0$.

We can use Eulers formula to get

$$\cos(at) = \frac{e^{iat}+e^{-iat}}{2}$$

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \mathcal{L}\left\{\frac{e^{iat}}{2}\right\} + \mathcal{L}\left\{\frac{e^{-iat}}{2}\right\} \\
&= \frac{1}{2}\frac{1}{s-ia} + \frac{1}{2}\frac{1}{s+ia} \\
&= \frac{s}{s^2+a^2}
\end{aligned}$$

Example 5: Let $u(t)$ be the unit step function. Then $u(t-a)$, $a > 0$ is the step function that is zero for $t < a$ and one when $t \geq a$. We have $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$.

$$\begin{aligned}
\mathcal{L}\{u(t-a)\} &= \int_0^\infty e^{-st}u(t-a)\,dt \\
&= \int_0^\infty e^{-st}u(t-a)\,dt \\
&= \int_a^\infty e^{-st}\,dt \\
&= \lim_{R\to\infty} \int_a^R e^{-st}\,dt \\
&= \lim_{R\to\infty} -\frac{e^{-sR}}{s} + \frac{e^{-as}}{s} \\
&= \frac{e^{-as}}{s}
\end{aligned}$$

Example 6:$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $s > 0$.

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty e^{-st}t^n)\,dt \\
&= \lim_{R\to\infty} \int_0^R e^{-st}t^n\,dt \\
&= \lim_{R\to\infty} -\frac{t^n e^{-st}}{s}\bigg|_0^R + \int_0^R \frac{ne^{-st}t^{n-1}}{s}\,dt
\end{aligned}$$

Here we used integration by parts. The left term evaluates to zero and only the integral is left. Repeated applications of integration by parts increases the power of s in the denominator and generates the n factorial in the numerator yielding a final result of $\frac{n!}{s^{n+1}}$.

Example 7: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ assuming $\lim_{t\to\infty} f(t)e^{-st} = 0$.

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}\,dt \\
&= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt \\
&= sF(s) - f(0)
\end{aligned}$$

Example 8: $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ assuming $\lim_{t\to\infty} f'(t)e^{-st} = 0$.

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= \int_0^\infty f''(t)e^{-st}\,dt \\
&= f'(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt \\
&= -f'(0) + s(\mathcal{L}f'(t) \\
&= -f'(0) + s(sF(s) - f(0)) \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}$$

The Inverse Laplace Transform(Sec 6.1.3)

If $F(s) = \mathcal{L}\{f(t)\}$ for some function $f(t)$, we define the inverse Laplace transform $\mathcal{L}^{-1}[F(s)] = f(t)$. There is an integral formula for the inverse, but it is beyond the scope of this course. For our purposes, the table of Laplace transforms and inverses will suffice.

Example 9: $\mathcal{L}^{-1}[\frac{1}{s+4}] = e^{-4t}$. This follows immediately from the fact that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

Often, we will need to perform an algebraic manipulation to $F(s)$ to put it in a nice form that matches the formulas given in the table of transforms.

Example 10: Find the Laplace inverse of $F(s) = \frac{s-1}{(s-2)(s+1)}$.

As written, $F(s)$ does not look like any of the transforms we know. However, we can apply partial fraction decomposition. Write

$$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

and solve for A, B. The easiest way to do this is to use what some people call the cover up method. Multiply both sides by $s-2$ and set $s = 2$ to isolate a.

$$\frac{s-1}{s+1}\Big|_{s=2} = a$$

Therefore, $a = \frac{1}{3}$. Similarly, multiply both sides by $s+1$ and set $s=-1$ to isolate b.

$$\left.\frac{s-1}{s-2}\right|_{s=-1} = b$$

Therefore, $b = \frac{2}{3}$. So we have that

$$\frac{s-1}{(s-2)(s+1)} = \frac{1}{3} \cdot \frac{1}{s-2} + \frac{2}{3} \cdot \frac{1}{s+1}$$

Now we can apply the same formula as in example 5 to get

$$\mathcal{L}^{-1}[\frac{s-1}{(s-2)(s+1)}] = \frac{1}{3} \cdot \mathcal{L}^{-1}[\frac{1}{s-2}] + \frac{2}{3} \cdot \mathcal{L}^{-1}[\frac{1}{s+1}] = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$