Algebra Prelim

January 2002

Provide proofs for all statements, citing any theorems that may be needed. If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.

In the following, $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{Z}$ denotes the ring of integers.

1. Let G be a group of order 49. We have made several statements about such a group. For each statement, **discuss with proper justification,** if the statement is true (for all possible groups of this order), or partially true (true only for some groups of this order) or false (for all groups of this order).

   (a) G contains an element of order 7.

   (b) G contains an element of order 49.

   (c) G contains an element of order 21.

   (d) G has non trivial center.

   (e) G is abelian.

   (f) G can be isomorphic to a subgroup of S_6- the symmetric group on 6 symbols.

2. (a) Recall the following theorem:

   If G is group and H is a subgroup of G having exactly m right cosets $\{H = g_1H, g_2H, \cdots, g_mH\}$. Then we have a homomorphism ϕ of the group G into S_m (thought of as the permutations of the m cosets) defined by $\phi(g)(g_iH) = gg_iH$.

   Prove that K, the kernel of this homomorphism is a subgroup of H.

   Explain why the kernel K is a normal subgroup of G as well as H.

(b) Using the above or otherwise, argue that a group of order 216 must have a proper normal subgroup (i.e. it is not simple).

3. Let M be a maximal ideal of $\mathbb{Z}[X]$ such that $M \bigcap \mathbb{Z} \neq 0$. Show that $M = (p, f)$, where p is a prime in $\mathbb{Z}$ and f is a monic polynomial in X of positive degree.

   Extra: Is it necessary to make the assumption about the intersection of M with $\mathbb{Z}$?

4. Let R be a commutative ring with $1 \neq 0$ and let Q be a proper ideal of R. Recall that Q is said to be a primary ideal, if

$$xy \in Q \ , \ x \notin Q \text{ implies } y^n \in Q \text{ for some positive integer } n.$$

   Prove that the radical $\sqrt{Q}$ of such a proper primary Q is prime.

5. Let R be a commutative ring with $1 \neq 0$. Let $D : R \longrightarrow R$ be a derivation (of R). (This means that D is an additive homomorphism which satisfies the product rule $D(xy) = xD(y) + yD(x)$ for all $x, y \in R$. Argue the following:

   (a) $Ker(D)$ is a subring of R containing 1.

   (b) For the ring $\mathbb{Z}[i]$ the zero map is the only derivation possible. (Here i stands for the complex square root of -1.)

6. Let S, T be subspaces of a vector space V over a field K. Argue that their union $S \bigcup T$ is also a subspace iff one of S, T is contained in the other.

7. Let $\phi : V \longrightarrow V$ be a linear transformation of a vector space V. Let $\lambda_1, \cdots, \lambda_n$ be distinct eigenvalues of ϕ and let $v_1, \cdots, v_n$ be eigenvectors belonging to them respectively. Then argue that $v_1, v_2, \cdots, v_n$ are linearly independent.

   Deduce that a linear transformation of an m-dimensional vector space into itself has at most m eigenvalues. Give an example of a three dimensional vector space and a linear transformation of it into itself which has only one eigenvalue such that its eigenspace (the space spanned by eigenvectors belonging to it) is one dimensional.

8. Calculate (with justification) the order of the Galois group of the polynomial $x^4 - 5$ over $\mathbb{Q}$.

9. What is the Galois group of the polynomial $x^4 - 5$ reduced mod 3, i.e. thought of as a polynomial over $\mathbb{Z}_3$?

   Hint: First carefully decide if the polynomial is reducible over $\mathbb{Z}_3$.