Algebra Prelim

January 2006

1. Prove that the symmetric group S_5 is generated by the two elements (1 2) and (1 2 3 4 5).

2. Let G be a group with $|G| = 200$. Show that G has a normal Sylow 5-subgroup which is abelian.

3. Let R be a ring (with identity) such that $a^2 = a$ for all $a \in R$. Prove that R is commutative.

4. Let $\mathbf{Q}$ be the field of rational numbers. Then:

   (a) Argue that $\mathbf{Q} \subset \mathbf{Q}(\sqrt{3}, \sqrt{2})$ is a Galois extension and find its Galois group.

   (b) Prove that $\mathbf{Q}(\sqrt{3}, \sqrt{2}) = \mathbf{Q}(\sqrt{2 - \sqrt{3}})$.

5. Determine all (up to isomorphism) rings S such that there is a surjective ring homonorphism
$\mathbf{R}[X]/(X^2+2)(X-3)(X-2) \to S$ Here $\mathbf{R}$ is the field of real numbers.

6. Let $k \subset L$ be a Galois extension where the degree of the extension is $17 \cdot 11 \cdot 7$. Argue that for every intermediate field K (so $k \subset K \subset L$) K is a Galois extension of k.

7. Let M consist of all the $n \times n$ matrices $A = (a_{ij})$ over the field $\mathbf{R}$ of real numbers whose trace is 0 (i.e. $\sum a_{ii} = 0$). Then:

   (a) Argue that M is a vector space over $\mathbf{R}$.

   (b) Find the dimension of M.

   (c) Decide if M is closed under the multiplication of matrices.

   (d) If $A, B \in M$ show that $AB - BA$ cannot be the identity matrix.

8. Let $f(X) = X^{12} - 1 \in \mathbf{Q}[X]$ (where $\mathbf{Q}$ is the field of rational numbers). Then:

   (a) Find the Galois group of the splitting field E of $f(X)$ over $\mathbf{Q}$.

   (b) Find all the subfields of E.

9. Let $R = \mathbf{Z}[\sqrt{-5}]$ and let K be the field of fractions of R. Show that the polynomial $f(X) = 3X^2 + 4X + 3$ is irreducible in $R[X]$ but that it is reducible in $K[X]$.