Algebra Prelim

January 7, 2013

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**
- **Problems (2^*) and (2^{**}) below correspond to different topics covered in the Linear Algebra course (MA 565) during either AY 2011-12 or AY 2012-13. Choose one of the two problems and clearly indicate which of the two you wish to be graded. Failure to do so will result in no credit being given for either of the two problems.**

Good Luck!

(1) Let p be a prime and consider $G = GL_2(\mathbb{Z}_p)$. That is, G consists of all 2×2 invertible matrices with coefficients in the field $\mathbb{Z}_p$. Find the number of elements of G.

As per the instructions above, please choose one of the following two problems.

(2^*) Let k be a field and n a positive integer. Consider the vector space k^n and the subspace $W := \{(\alpha_1, \ldots, \alpha_n) \in k^n \mid \alpha_1 + \cdots + \alpha_n = 0\}$.

a) Determine the dimension of W?

b) Recall that the annihilator of W, denoted with W^0, consists of all linear functionals on k^n that vanish on W, that is, $W^0 = \{f \in (k^n)^* \mid f(w) = 0 \text{ for all } w \in W\}$. Show that W^0 is a subspace of $(k^n)^*$. What is the dimension of W^0?

c) Prove that W^0 consists of all linear functionals f of the form

$$f(\alpha_1, \ldots, \alpha_n) = c \sum_{j=1}^{n} \alpha_j \text{ for all } (\alpha_1, \ldots, \alpha_n) \in W,$$

for some $c \in k$.

(2^{**}) Let V be a vector space over the field k and $T : V \to V$ be a linear map. Suppose that T has the property that $\operatorname{Im}(T) = \{v \in V \mid T(v) = v\}$.

a) Show that $T \circ T = T$.

b) Show that $\operatorname{Ker}(T) \oplus \operatorname{Im}(T) = V$.

c) Show that T is diagonalizable and describe a diagonal matrix representation as precisely as possible.

(3) Let G be a group of odd order and let H be a normal subgroup of order 5. Show that H is contained in the center of G.

[**Hint:** Consider the centralizer $C_G(H)$ and show that $G/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.]

(4) Let p be a prime number and let G be a group of order p^n for some $n \in \mathbb{Z}^+$. Suppose that G acts on a finite set S. Show that

$$|S| \equiv |S_0| \mod p,$$

where $S_0 = \{x \in S \mid g \cdot x = x \text{ for all } g \in G\}$.

(5) Let R be a commutative ring with identity. Suppose that for each $a \in R$ there exists a natural number $n \geq 2$ such that $a^n = a$. (Observe that n can depend on a.) Show that every prime ideal of R is a maximal ideal.

(6) Show that the domain $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

(7) Let k be a field of characteristic $p > 0$ and let $a \in k$. Consider the polynomial $f = X^p - X - a \in k[X]$.

a) Show that $f(c) = f(c+i)$ for any c in an extension field of k and for any $i \in \mathbb{Z}$.

b) Suppose c is a root of f in a splitting field E of k. Determine the prime factorization of f in $E[X]$.

c) Show that f is irreducible in $k[X]$ if and only if f has no root in k.

[**Hint:** For one implication consider a typical non-trivial factor of f in $E[X]$ and determine its second highest coefficient.]

(8) Let $n \geq 3$ and let ζ be a primitive n-th root of unity over $\mathbb{Q}$. Recall that $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(n)$, where φ is the Euler φ-function. Prove that $\alpha := \zeta + \zeta^{-1}$ is algebraic over $\mathbb{Q}$ of degree $\varphi(n)/2$.

[**Hint:** It will be useful to note that $\alpha \in \mathbb{R}$. If you want to use this fact then you also have to prove it.]

(9) Show that there exists a Galois extension of $\mathbb{Q}$ whose Galois group is $\mathbb{Z}_7$.

[**Hint:** Start with a suitable primitive root of unity.]