Algebra Prelim, January 6, 2016

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) In the real vector space $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuously differentiable}\}$ consider the subspace $V := \langle e_1, e_2, e_3, e_4 \rangle$, where

$$e_1(x) = e^x, \quad e_2(x) = e^{2x}, \quad e_3(x) = \sin(x), \quad e_4(x) = \cos(x).$$

Then $\mathcal{A} := \{e_1, e_2, e_3, e_4\}$ forms a basis of V (you need not show this). Consider the linear map

$$T : V \longrightarrow V, \quad f \longmapsto f' \quad \text{(the derivative of } f\text{)}.$$

a) Give the matrix representation of T with respect to the basis $\mathcal{A}$.

b) Determine all eigenvalues of T in $\mathbb{R}$.

c) For each eigenvalue determine the corresponding eigenspace of T. Make sure your answer is a subspace of V (i.e., it consists of functions).

d) Is T diagonalizable over $\mathbb{R}$?

e) Is T triangulable over $\mathbb{R}$?

(2) Let V be a finite-dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) with inner product denoted by $\langle \cdot, \cdot \rangle$. Let T be a self-adjoint linear map on V, that is,

$$\langle v, T(w) \rangle = \langle T(v), w \rangle \text{ for all } v, w \in V.$$

As usual, the powers of T are defined as $T^k := T \circ \cdots \circ T$ (k times). Show

$$T \text{ nilpotent} \Longrightarrow T = 0.$$

(3) Let G be a finite group and $N \trianglelefteq G$ be a normal subgroup of G. Let p be a prime divisor of $|N|$ and suppose N has a unique p-Sylow subgroup.

a) Suppose p does not divide $[G : N]$. Show that G has a unique p-Sylow subgroup.

b) Suppose p divides $[G : N]$. Give an example where the conclusion from (a) does not hold.

[Hint: Consider, for example, a group of order 12.]

(4) Let $n \geq 5$ and let A_n denote the alternating group on n symbols.

a) Let $G \leq A_n$ be a subgroup such that $[A_n : G] < n$. Show that $G = A_n$.

[Hint: Recall that A_n acts on the set of left cosets of G.]

b) Is there a subgroup $H \leq A_n$ such that $[A_n : H] = n$? Support your answer.

(5) Let

$$\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[x] \mid f(m) \in \mathbb{Z} \text{ for all } m \in \mathbb{N}\}.$$

You do not have to prove that $\mathrm{Int}(\mathbb{Z})$ is a subring of $\mathbb{Q}[x]$, but make sure you understand this. Note that for all $n \in \mathbb{N}$ the polynomial $\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!}$ is in $\mathrm{Int}(\mathbb{Z})$ because binomial coefficients $\binom{m}{n}$ are integers.

a) Determine the group of units of $\mathrm{Int}(\mathbb{Z})$.

b) Show that 2 is irreducible but not prime in the ring $\mathrm{Int}(\mathbb{Z})$.

(6) For which $n \in \mathbb{N}$ is the polynomial $\sum_{i=0}^{n} x^i \in \mathbb{Q}[x]$ irreducible? Support your answer.

(7) Let $K \subseteq \mathbb{C}$ be a subfield such that $K \mid \mathbb{Q}$ is Galois with cyclic Galois group of order 4.

a) Show that K has a unique subfield L such that $[L : \mathbb{Q}] = 2$.

b) Show that $\sigma(K) \subseteq K$, where σ denotes complex conjugation.

c) Show that the subfield L in part (a) is contained in $\mathbb{R}$.

(8) Let q be a prime power and $m \in \mathbb{N}$. Consider the finite fields $\mathbb{F}_q \subset \mathbb{F}_{q^m}$ and the map

$$\tau : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_{q^m}, \quad a \longmapsto \sum_{i=0}^{m-1} a^{q^i}.$$

Show the following.

a) τ is $\mathbb{F}_q$-linear.

b) $\mathrm{im}\,(\tau) \subseteq \mathbb{F}_q$, where $\mathrm{im}\,(\tau)$ denotes the image of the map τ.

c) τ is not the zero map.

d) $\mathrm{im}\,(\tau) = \mathbb{F}_q$.

(9) Let $K \subseteq \mathbb{C}$ be the splitting field of $f = x^5 - 2$ over $\mathbb{Q}$.

a) Show that $[K : \mathbb{Q}] = 20$.

b) Show that there exists a unique subfield L of K such that $[K : L] = 5$.

c) Give the subfield L explicitly.