Algebra Prelim, January, 2026

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Consider the following matrix.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a) Find the characteristic polynomial of A.

b) Let $I \subset \mathbb{C}[x]$ be the ideal of polynomials $p(x)$ with the property that $p(A) = 0$. Find a polynomial that generates I.

(2) Let V be a vector space of dimension n, and let $T : V \to V$ be a linear transformation with $T^2 = 0$.

a) Show that $\mathrm{Im}(T) \subseteq \mathrm{Ker}(T)$.

b) Show that the rank of T is at most $\frac{n}{2}$.

(3) Let G be a group of order 105.

a) Show that G must have a normal Sylow 5-group or a normal Sylow 7-group.

b) Prove that G has a normal subgroup of order 35.

(4) Let G be a finite group and let $H \subset G$ be a subgroup with $[G : H] = p$, where p is the smallest prime dividing the order of G. Prove that H is a normal subgroup of G.

(5) Let R be a commutative ring with identity.

a) Define what it means for an ideal of R to be maximal.

b) Define what it means for an ideal of R to be prime.

c) Prove that if R is a principal ideal domain (PID) then a nonzero ideal is prime if and only if it is maximal.

(6) Consider the ideal $I = \langle 5, x^2 + 2 \rangle$ in the ring $\mathbb{Z}[x]$. Prove that $\mathbb{Z}[x]/I$ is a field.

(7) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

a) Determine $[K : \mathbb{Q}]$.
b) Find $\mathrm{Gal}(K/\mathbb{Q})$ and all intermediate subfields between $\mathbb{Q}$ and K.

(8) Let $f(x) = x^4 - 5$.

a) Find the splitting field K of $f(x)$ over $\mathbb{Q}$.
b) Show that $\mathrm{Gal}(K/\mathbb{Q})$ is the dihedral group D_8 (the symmetries of a square).

(9) Let F be a finite field of characteristic p. Recall that the Frobenius map $\Phi : F \to F$ is the map which takes an element $a \in F$ to a^p.

a) Show that Φ is a field automorphism.
b) Show that every element in F has a p-th root.