Algebra Prelim

June 2003

Provide proofs for all statements, citing any theorems that may be needed.
If necessary, you may use the results from other parts of this test, even though you may not have successfully proved them.
Note the notations:
$\mathbb{C}$ = **the field of complex numbers,** $\mathbb{R}$ = **the field of reals,**
$\mathbb{Q}$ = **the field of rationals,** $\mathbb{Z}$ = **the ring of integers, and**
$\mathbb{N}$ = **the set of natural numbers.**

1. Consider the set V of 3×3 matrices (r_{ij}) over $\mathbb{R}$ such that $r_{11} = -r_{21}$ and such that $r_{33} = r_{22}$. Argue that this set V is a vector space over $\mathbb{R}$ and determine its dimension.

2. Determine up to isomorphism all rings R such that there is a surjective ring homomorphism $\mathbb{Z}/15\mathbb{Z} \to R$.

3. Argue that the multiplicative group $\mathbb{C}^\times$ of the complex numbers has a unique subgroup of order $n \in \mathbb{N}$ for all $n \geq 1$.

4. Consider the ideals $(X, 2)$ and $(X^2 + X, X^2 + 2X + 1)$ in $\mathbb{Z}[X]$. Decide if these are principal ideals. Justify your answer.

5. Let $\alpha := \sqrt[3]{5} \in \mathbb{R}$. Determine the minimal polynomial of α over $\mathbb{Q}$ and decide if the Galois group of the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is commutative. Justify your answer.

6. Let k be a field with 64 elements.
   (a) How many subfields does k have? Describe them.
   (b) What is $\dim_{\mathbb{Z}/2\mathbb{Z}}(k)$?
   (c) How many subgroups does the multiplicative group $k^\times$ of k have? Describe them.

7. Prove that
$$(\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \not\cong (\mathbb{Z}/16\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}).$$

8. (a) Let G and H be finite groups. Suppose that G has s conjugacy classes and that H has t conjugacy classes. Argue that $G \times H$ has st conjugacy classes.
   (b) Let G and H be simple groups. Find all the normal subgroups of $G \times H$.