Algebra Prelim

June 2, 2010

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

1. Let V be the subspace of the $\mathbb{R}$-vector space of differentiable functions $\mathbb{R} \to \mathbb{R}$ that is generated by $f, \sin, \cos$, and $\exp$, where $f(x) = 1$ and $\exp(x) = e^x$ for all $x \in \mathbb{R}$. Consider the linear map

$$\varphi : V \to V,\ g \mapsto g + g'.$$

(a) Determine the presentation matrix of φ with respect to the basis $\{f, \sin, \cos, \exp\}$. (You may use without proof that this is indeed a basis of V.)

(b) Find all real eigenvalues of φ.

(c) Determine a basis of the eigenspace of φ to each real eigenvalue.

2. Let A be an orthogonal matrix, that is, a square matrix with real entries such that $A \cdot A^T$ is the identity matrix. Show:

(a) If v and w are eigenvectors of A to the eigenvalues 1 and -1, respectively, then $v^T \cdot w = 0$.

(b) The matrix A is diagonalizable over the real numbers if and only if its minimal polynomial over $\mathbb{R}$ divides the polynomial $(X-1)(X+1)$.

3. Let H be a subgroup of a group G, and let $g \in G$ be any element. Assume that the right coset Hg equals *some* left coset of H in G. Show that then $gH = Hg$.

4. Let a, b be elements of finite order in a group G such that $ab = ba$ and $\langle a \rangle \cap \langle b \rangle = \{e\}$, where e denotes the identity of G. Show that the order of ab is the positive least common multiple of the orders of a and b.

5. Let E/K be a Galois extension of degree 200. Prove that E has a unique subfield L containing K such that L/K is a Galois extension of degree 8.

6. Find a generator of the ideal I of the polynomial ring $\mathbb{F}_5[X]$ that is generated by $X^4 + 4X^3 + 2X + 4$ and $X^3 + 3X^2 + 3$ (in other words, find a generator of the principal ideal I).

7. Let $K[X]$ be the polynomial ring over a field K, and consider

$$R := \{f \in K[X] \mid f'(0) = 0\},$$

where f' denotes the derivative of f. Show:

(a) R is a subring of $K[X]$, thus it is an integral domain.

(b) The elements X^2 and X^3 are irreducible in R.

(c) R is not a factorial domain.

8. Let $\alpha := \sqrt[3]{5} + 1 \in \mathbb{R}$.

(a) Find the minimal polynomial f of α over $\mathbb{Q}$.

(b) Argue that α is the only real root of f. (Hint: Consider $f(X+1)$.)

(c) Determine the automorphism group of the field $\mathbb{Q}(\alpha)$.

9. Let $E \subset \mathbb{C}$ be the splitting field of $X^4 + 1$ over $\mathbb{Q}$. Determine explicitly all automorphisms and subfields of E.

10. Let E/K be a Galois extension with Galois group G. Suppose there is an element $\alpha \in E$ such that $\sigma(\alpha) \neq \tau(\alpha)$ for all distinct automorphisms $\sigma, \tau \in G$. Prove that $E = K(\alpha)$. (Hint: Consider the Galois group of $E/K(\alpha)$.)