Algebra Prelim, June 6, 2017

- **Provide proofs for all statements, citing theorems that may be needed.**
- **If necessary you may use the results from other parts of this test, even though you may not have successfully proved them.**
- **Do as many problems as you can and present your solutions as carefully as possible.**

Good luck!

(1) Let $n \in \mathbb{N}$, and let K be a field whose characteristic does not divide n. Consider the $n \times n$ matrix with entries in K

$$A = \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

a) Show that A is idempotent, that is, $A^2 = A$.

b) Find an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix. Specify S and the diagonal matrix explicitly.

(2) Let $\varphi, \psi : V \to V$ be endomorphisms (linear maps) on a finite-dimensional vector space over a field K such that $\varphi \circ \psi = \psi \circ \varphi$. Denote the identity map on V by id_V.

a) For every $\mu \in K$, show that $(\mu\,\mathrm{id}_V - \varphi)$ maps eigenvectors of ψ onto eigenvectors of ψ.

b) Assume V has a basis consisting of eigenvectors of ψ and a basis consisting of eigenvectors of φ. Prove that V has a basis consisting of vectors that are eigenvectors of both φ and ψ.

(3) Consider the symmetric group S_8 on 8 elements.

a) How many elements of S_8 can be written as a disjoint product of a 4-cycle and a 2-cycle?

b) Determine the number of elements in S_8 that have order four.

(4) Let H be a subgroup of a group G with $[G : H] = n$. Prove that there is a normal subgroup N of G with $N \subset H$ such that $[G : N] \leq n!$.

(5) Let K be a field, and consider the set $R = \{f \in K[x] \mid f'(1) = f''(1) = 0\}$. Show:

a) R is a subring of $K[x]$.

b) $(x-1)^3$ and $(x-1)^4$ are irreducible elements of R.

c) R is not a UFD (aka factorial domain).

(6) Fix a prime number p, and consider the subset

$$R = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ coprime}, p \text{ does not divide } b \right\} \subset \mathbb{Q}.$$

a) Show that R is a subring of $\mathbb{Q}$.

b) Determine the units of R.

c) Prove that each nonzero ideal of R is principal and equal to (p^e) for some integer $e \geq 0$.

d) Describe explicitly the quotient field of R as a subfield of $\mathbb{Q}$.

(7) Let t be a variable. Show that the field extension $\mathbb{C}(t)/\mathbb{C}(t^{20})$ is Galois and determine the isomorphism type of its Galois group.

(8) Let K be the splitting field of the polynomial $f = x^4 + x^3 + 1$ over $\mathbb{F}_2$.

a) Determine K up to isomorphism.

b) Find the least $m \in \mathbb{N}$ such that f divides $x^m - 1$ in $\mathbb{F}_2[x]$.

c) Let α be a root of f. Describe explicitly all roots of f.

(9) a) Determine the Galois group of the polynomial $x^3 - 5$ over $\mathbb{Q}$ explicitly.

b) Show that the real number $\sqrt[3]{5}$ is not contained in any cyclotomic field extension of the rational numbers.