ALGEBRA PRELIM, JUNE 2026

- **Provide proofs of all statements, citing theorems that may be needed.**
- **Do as many problems as possible and present your solutions as carefully as possible.**
- **All problems carry the same weight, but the individual parts of a problem may have different weights.**

(1) Let k be a field, let $a_0, a_1, \ldots, a_d$ be distinct elements of k, and define the polynomial

$$p_j(x) = \prod_{i \neq j} (x - a_i).$$

Show that $\{p_0(x), \ldots, p_d(x)\}$ is a basis for the vector space of polynomials of degree at most d with coefficients in k.

(2) Let V be a finite-dimensional vector space over a field F. Let $S, T \colon V \to V$ be linear maps satisfying $ST = TS$, and assume both S and T are diagonalizable over F.

(a) Show that each eigenspace of T is S-invariant.

(b) Prove that V has a basis consisting of vectors that are eigenvectors for both S and T.

(c) Deduce that ST is diagonalizable.

(3) Let G, H be finite groups. Prove that $G \times H$ is cyclic if and only if both G and H are cyclic and the orders of G and H are relatively prime.

(4) Let $\mathrm{SL}_2(\mathbb{F}_q)$ be the group of 2×2 matrices with determinant 1 and entries in $\mathbb{F}_q$.

(a) Consider the action of $\mathrm{SL}_2(\mathbb{F}_q)$ on the set of 1-dimensional subspaces of $\mathbb{F}_q^2$. Find the stabilizer of $V = \{(x, y) \in \mathbb{F}_q^2 \mid y = 0\}$.

(b) Use the orbit-stabilizer theorem to compute the order of $\mathrm{SL}_2(\mathbb{F}_q)$.

(5) Show that there is no ring homomorphism $f\colon \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{3}]$ such that $f(1) = 1$.

(6) Let R be a commutative ring with identity, and let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x].$$

(a) Suppose first that R is an integral domain. Prove that $f(x)$ is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_i = 0$ for all $i \geq 1$.

(b) Now let R be an arbitrary commutative ring with identity. Prove that $f(x)$ is a unit in $R[x]$ if and only if a_0 is a unit in R and a_i is nilpotent for all $i \geq 1$. (You may use the fact that an element $a \in R$ is nilpotent if and only if it is contained in all prime ideals $\mathfrak{p} \subseteq R$.)

(7) (a) Is $\mathbb{F}_{25}$ isomorphic to a subfield of $\mathbb{F}_{125}$?

(b) Consider the polynomial $F_k(x) = x^{p^k} - x \in \mathbb{F}_p[x]$. Show that the degree of any irreducible factor of $F_k(x)$ divides k.

(8) Prove that no finite field is algebraically closed.

(9) Let $f(x) \in \mathbb{Z}[x]$ be an irreducible quartic with Galois group S_4 over $\mathbb{Q}$, and let α be a root of $f(x)$. Show that $\mathbb{Q}$ is the only field that is properly contained in $\mathbb{Q}(\alpha)$.