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ANALYSIS PRELIMINARY EXAM

January 7, 2011

NAME: ______________________________ OPTION: ______________

Instructions

1. This is a three hour examination which consists of two parts: **Advanced Calculus and Real** *or* **Complex Analysis.** You must work problems from the section on Advanced Calculus and from the section of the option, Real or Complex, that you have chosen.

2. You need to work a total of 5 problems: Four mandatory problems (two mandatory problems from each of two parts), and one optional problem from either advanced calculus or the option you have chosen. Please indicate clearly on your test paper whether you are taking the **Real or Complex Analysis option,** and which optional problem you are solving. **Only 5 problems will be graded.**

3. Do not put your name on any sheet except the cover page. The papers will be blind-graded.

4. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to partial solutions of the easy parts of two different problems. Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Let $\{f_n\}$ be a sequence of continuous, real-valued functions on $[0,1]$, and f be a real-valued function on $[0,1]$.

   (a) Show that, if $f_n \to f$ uniformly on $[0,1]$, then $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$.

   (b) Give an example for which $f_n \to f$ pointwise on $[0,1]$, but $\int_0^1 f_n(x)\,dx \not\to \int_0^1 f(x)\,dx$.

2. Suppose that f is a real-valued function on $[a,b]$ which is twice differentiable at some $c \in (a,b)$.

   (a) Prove that
   $$f''(c) = \lim_{h\to 0} \frac{f(c+h)+f(c-h)-2f(c)}{h^2} \; .$$

   (b) If we further assume that $f(tx+(1-t)y) < tf(x)+(1-t)f(y)$ for all $x,y \in [a,b]$ and $t \in (0,1)$ (in other words, f is convex on $[a,b]$), prove that $f''(c) \geq 0$.

Advanced Calculus, Optional Problems

1. (a) Let $f : E \to E$, where E is a closed subset of $\mathbb{R}$ (note that we are not assuming that E is connected), and assume that $|f(x)-f(y)| \leq \lambda|x-y|$ for all $x,y \in E$, where $\lambda \in (0,1)$. Show that there is a unique point $z \in E$ for which $f(z) = z$.

   (b) Suppose that f is differentiable on $E = (0,1)$. Show that $|f(x) - f(y)| \leq \lambda|x-y|$ for all $x,y \in E$ if and only if $|f'(x)| \leq \lambda$ for all $x \in E$.

2. (a) Suppose that f is a nonconstant, real-valued and continuous function on $[a,b]$. Prove that, if f does not change sign on $[a,b]$, then $\int_a^b f(x)\,dx \neq 0$.

   (b) Suppose that p is a polynomial of degree n with real coefficients, and that
   $$\int_a^b p(x)\,x^k\,dx = 0 \quad \text{for} \quad k = 0,1,2,\ldots,n-1 \; .$$
   Show that the n roots of p are: real, distinct, and lie within the open interval (a,b). Hint: Any degree-n polynomial q may be written as $q(x) = \alpha\prod_{j=1}^{n}(x-r_j)$, where α is the leading coefficient of q and $\{r_j\}$ are its (possibly non-distinct) roots; if the coefficients of q are real, then so is α, and any complex roots of q come in conjugate pairs.

Real Analysis, Mandatory Problems

1. (a) State the Lebesgue's Dominated Convergence Theorem.
   (b) State the Fatou's Lemma for nonnegative functions.
   (c) Use the Fatou's Lemma in part (b) to prove the Lebesgue's Dominated Convergence Theorem for nonnegative functions.

2. Show that if f is Lebesgue integrable on $\mathbb{R}^n$, then

$$\alpha|\{x \in \mathbb{R}^n : \ |f(x)| > \alpha\}| \to 0$$

as $\alpha \to \infty$.

Real Analysis, Optional Problems

1. Prove or disprove the following statement. If $f, g : [0,1] \to [0,1]$ are measurable, then $h(x) = f(g(x))$ is measurable.

2. Let f be a measurable function on $[0,1]$. Suppose that

$$M = \inf\{\alpha : |\{x \in [0,1] : \ |f(x)| > \alpha\}| = 0\} < \infty.$$

Show that $|f(x)| \leq M$ for a.e. $x \in [0,1]$ and

$$\lim_{p\to\infty} \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{1/p} = M.$$

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Complex Analysis, Mandatory Problems

1. Show that

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{3}.$$

2. Let $F(z)$ be an entire function with the property that

$$|F(z)| \leq C|z|^5 \qquad (*)$$

for some fixed constant C whenever $|z| \geq R$.

(a) Prove that F is a polynomial of degree ≤ 5.

(b) If the inequality (*) holds for all $z \in \mathbb{C}$, what more can be inferred? Explain!

Complex Analysis, Optional Problems

1. Let $P(z)$ be a polynomial of degree ≥ 2. Prove that

$$\sum_{\mathbb{C}} Res \frac{1}{P(z)} = 0.$$

2. Prove that there is no function $f(z)$ analytic in the open unit disk $|z| < 1$ with the property that

$$f(\frac{1}{n}) = \frac{(-1)^n}{n^2}, \quad n = 2, 3, 4, ...$$