Preliminary Examination in Analysis

January 2014

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Let f be differentiable at a and let $\{x_n\}$, $\{z_n\}$ be two sequences converging to a such that $x_n < a < z_n$ for all $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \frac{f(x_n) - f(z_n)}{x_n - z_n} = f'(a).$$

2. Assume that f_n, $n \in \mathbb{N}$, are monotone on $[a, b]$. Show that if $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely at $x = a$ and $x = b$, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on the whole interval $[a, b]$.

Advanced Calculus, Optional Problems

3. Suppose that $f, g \in C([a, b])$. Show that there is $\theta \in (a, b)$ such that

$$g(\theta) \int_a^b f(x)\,dx = f(\theta) \int_a^b g(x)\,dx.$$

4. Let $\{f_n\}$ be a sequence of nonnegative continuous functions on the interval $I = [-1, 1]$ and suppose $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for each $x \in I$. Prove or disprove the assertion that if f is continuous on I, then the series for f converges uniformly on I.

Real Analysis, Mandatory Problems

1. Let E be a subset of $\mathbb{R}^d$.
(a) State the definition of Lebesgue exterior measure $m_*(E)$.
(b) Prove that the exterior measure is translation invariant; that is, if

$$E_h = \{x + h : \; x \in E\},$$

where $h \in \mathbb{R}^d$, then $m_*(E_h) = m_*(E)$.

2. Let $f(x, y) : 0 \le x, y \le 1$ satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $\frac{\partial f}{\partial x}(x, y)$ is a bounded function of (x, y). Show that $\frac{\partial f}{\partial x}(x, y)$ is a measurable function of y for each x, and

$$\frac{d}{dx}\int_0^1 f(x, y)\,dy = \int_0^1 \frac{\partial f}{\partial x}(x, y)\,dy.$$

Real Analysis, Optional Problems

3. Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \le f$ a.e. on E, show that

$$\int_E f_k \to \int_E f.$$

4. Let $F \subset \mathbb{R}$ be a closed set such that $m(F^c) < \infty$ and

$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x - y|^2}\,dy,$$

where $\delta(y) = d(y, F) = \inf\{|y - z| : z \in F\}$. Show that $I(x) < \infty$ for a.e. $x \in F$.