ANALYSIS PRELIMINARY EXAM
JANUARY 12, 2015

Instructions

1. This is a three hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis. You must work problems from the Advanced Calculus section and the Real Analysis section or the Complex Analysis section, depending on the option you choose.

2. You should attempt a total of five problems: four mandatory problems (two from each section) and one optional problem from either section. Please indicate clearly whether you are taking the Real Analysis option or the Complex Analysis option and which optional problem is to be graded. If you do not indicate which optional problem is to be graded, the one with the lowest score will be used to determine your grade .

3. Do not put your name on any sheet except the cover sheet. The exam will be blind graded.

4. Each question is weighted equally.

5. You should provide complete and detailed solutions to each problem that you work. More weight will be given for a complete solution to one problem than for solutions to the easy bits in two different problems.

6. Indicate clearly the definitions and theorems you are using.

ADVANCED CALCULUS
MANDATORY PROBLEMS

1. (a) Show that the formula

$$f(x) = \lim_{m\to\infty} \left(\lim_{n\to\infty} \cos^{2n}(m!\pi x) \right)$$

defines a function on $[0, 1]$. Determine $f(x)$ for each $x \in [0, 1]$.

(b) Is the function f Riemann-integrable on $[0,\ 1]$? Justify your answer. Evaluate the Riemann integral $\int_0^1 f(x)\, dx$ if it exists.

2. Let $g : [0, 1) \to \mathbf{R}$ be uniformly continuous. Prove that there is a unique continuous function $h : [0, 1] \to \mathbf{R}$ such that $h(x) = g(x)$ for each $x \in [0, 1)$.

OPTIONAL PROBLEMS

3. Consider a sequence of Riemann-integrable functions $f_n : [0, 1] \to \mathbf{R}$ $(n = 1, 2, 3, \cdots)$ which, for some $M > 0$, satisfy

$$|f_n(x)| \leq M \qquad \text{for each } n \text{ and each } x \in [0, 1].$$

For each n, let

$$F_n(x) = \int_0^x f_n(t)\, dt.$$

Prove that the sequence $\{F_n\}$ has a subsequence which converges uniformly on $[0, 1]$.

4. Let $f : [0, 1] \to [0, 1]$ be a continuous function that satisfies

$$|f(x) - f(y)| < |x - y| \qquad \text{for } x \neq y \in [0, 1].$$

Prove that there is a unique $x_0 \in [0, 1]$ which satisfies $f(x_0) = x_0$.

REAL ANALYSIS
MANDATORY PROBLEMS

If $E \subset \mathbf{R}$ is a Lebesgue measurable set let $m(E)$ denote the Lebesgue measure of E.

1. Let $(E_n), n = 1, 2, \ldots,$ be a sequence of Lebesgue measurable sets with $E_{n+1} \subset E_n \subset [0, 1]$ for $n = 1, 2, \ldots$ Show that

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{k \to \infty} m(E_k).$$

2. Let f be an integrable function on $[0, 1]$. Given $\epsilon > 0$ show there exists $\delta > 0$ such that the following statement is true. If $E \subset [0, 1]$ is Lebesgue measurable with $m(E) \leq \delta$, then

$$\int_E |f| \, dx \leq \epsilon.$$

OPTIONAL PROBLEMS

3. Let (f_n) be a sequence of integrable functions on [0,1]. Suppose there exists an integrable function f on $[0, 1]$ with

$$\lim_{n \to \infty} \int_0^1 |f_n(x) - f(x)| \, dx = 0.$$

Show there exists a subsequence (f_{n_k}) of (f_n) with $\lim_{k \to \infty} f_{n_k}(x) = f(x)$ for Lebesgue almost every $x \in [0, 1]$.

4. Show that if f is integrable on [0,1] then

$$\lim_{k \to \infty} \int_0^1 f(x) \sin(kx) \, dx = 0.$$

COMPLEX ANALYSIS
MANDATORY PROBLEMS

1. Use the residue theorem to verify that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n\,\sin(\pi/n)} \quad \text{whenever } n = 2, 3, 4, \ldots$$

2. If f is a nonconstant entire function, prove that the range of f is dense in $\mathbb{C}$.

OPTIONAL PROBLEMS

3. Suppose that F is analytic in $|z| < 1$, continuous on $|z| \le 1$, and $|F(z)| \le M$ on $|z| \le 1$.

(i) If $F(\alpha_1) = F(\alpha_2) = \cdots = F(\alpha_k) = 0$, where $|\alpha_j| < 1$ for each $j = 1, 2, \ldots, k$ prove that

$$|F(z)| \le M \left| \left(\frac{z-\alpha_1}{1-\overline{\alpha}_1 z}\right)\left(\frac{z-\alpha_2}{1-\overline{\alpha}_2 z}\right)\cdots\left(\frac{z-\alpha_k}{1-\overline{\alpha}_k z}\right) \right|$$

whenever $|z| < 1$, and deduce that $|\alpha_1 \cdot \alpha_2 \cdots \alpha_k| \ge \dfrac{|F(0)|}{M}$.

(ii) If $F(0) \neq 0$ prove that the number of zeros of F in the disk $|z| \le 1/4$ does not exceed

$$\frac{1}{\log 4} \log \left| \frac{M}{F(0)} \right|.$$

4. Suppose that a function f is defined and analytic in the entire complex plane $\mathbb{C}$, and that for each $z_0 \in \mathbb{C}$ at least one coefficient c_n in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.