# Preliminary Examination in Analysis

**January 2021**

**Instructions**

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option that you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work the two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly which theorems and definitions you are using.

## Advanced Calculus, Mandatory Problems

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $\lim_{x\to\infty} f(x) = 1$. Show that

$$\lim_{b\to\infty} \frac{1}{b} \int_0^b f(x)\, dx = 1.$$

2. Suppose $A, B \subset \mathbb{R}$ are nonempty and bounded above, and define

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

Show that $\sup(A + B) = \sup A + \sup B$.

## Advanced Calculus, Optional Problems

3. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Show that for any $\epsilon > 0$, there exists $M > 0$ such that

$$|f(x) - f(y)| < M|x - y| + \epsilon$$

for all $x, y \in [0, 1]$.

4. Suppose $f$ is positive and differentiable on $\mathbb{R}$, and that $f'$ is bounded. Show that if $\int_0^\infty f(x)\, dx$ is finite then $\lim_{x\to\infty} f(x) = 0$.

## Real Analysis, Mandatory Problems

1. (a). Let $f$ be a real-valued function in $\mathbb{R}^d$. Define what it means for $f$ to be (Lebesgue) measurable on $\mathbb{R}^d$.

(b). Let $f$ and $g$ be two real-valued functions on $\mathbb{R}^d$. Suppose that $f$ is measurable on $\mathbb{R}^d$ and $g = f$ a.e. in $\mathbb{R}^d$. Use the definition in part (a) to show that $g$ is measurable.

(c). Let $\{f_k\}$ be a sequence of real-valued measurable functions on $\mathbb{R}^d$. Suppose that $\lim_{k\to\infty} f_k(x)$ exists for a.e. $x \in \mathbb{R}^d$. Show that $g(x) := \lim_{k\to\infty} f_k(x)$ is measurable.

2. Let $f$ be a Lebesgue integrable function in $\mathbb{R}^d$. Define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x)\,dx$$

for $\xi \in \mathbb{R}^d$. Show that $\widehat{f}$ is a bounded continuous function in $\mathbb{R}^d$.

## Real Analysis, Optional Problems

3. Let $\{f_k\}$ be a sequence of Lebesgue measurable functions defined on a measurable set $E \subset \mathbb{R}^d$ with $m(E) < \infty$. Suppose that for each $x \in E$,

$$M_x := \sup\left\{|f_k(x)| :\ k \geq 1\right\} < \infty.$$

Show that for each $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subset E$ such that $m(E \setminus F_\varepsilon) < \varepsilon$ and

$$M := \sup\left\{|f_k(x)| :\ x \in F_\varepsilon \text{ and } k \geq 1\right\} < \infty.$$

4. Let $f$ be a Lebesgue integrable function on $\mathbb{R}$. Prove that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_E |f(x)|\,dx < \varepsilon \quad \text{whenever } E \subset \mathbb{R} \text{ is measurable and } m(E) < \delta.$$

## Complex Analysis, Mandatory Problems

1. Compute the integral

$$\int_0^\infty \frac{x^2}{1+x^4}\, dx.$$

Provide all the details of your calculation.

2. Let $f : \mathcal{A} \to \mathbb{C}$ be analytic and suppose $B_R(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < R\} \subset \mathcal{A}$. Let $S_n(z)$ be the $n^{\text{th}}$-partial sum of the Taylor expansion of $f$ about $z_0$. Prove that there exists $M_f \geq 0$ so that for all $0 \leq r < R$,

$$|f(z) - S_n(z)| \leq M_f \left(\frac{r}{R}\right)^{n+1} \frac{1}{1 - \left(\frac{r}{R}\right)},$$

if $|z - z_0| \leq r$. Provide an explicit formula for $M_f$ showing how it depends on $f$, $z_0$, and the parameters $r, R$.

## Complex Analysis, Optional Problems

3. Find all entire functions $f(z)$ satisfying the bound

$$|f(z)| \geq |z|,$$

for all $z \in \mathbb{C}$.

4. This is a problem on Laurent expansions.
   (a). State a theorem on the existence and structure of the Laurent expansion of a function $f$ in an annular region about $z_0$ given by $\{z \in \mathbb{C} \mid 0 \leq r < |z - z_0| < R\}$, for $0 \leq r < R \leq \infty$.
   (b). Suppose a function $f$ is analytic on the annulus about $z_0 = 0$ with $r = 0$ and $R = 1$. If $f$ satisfies the bound

$$|f(z)| \leq \frac{1}{|z|^{\frac{1}{2}}},$$

   then prove that $f$ has a removable singularity at $z_0 = 0$ and that $f$ extends to an analytic function $g$ on the disk $|z| < 1$ with $|g(z)| \leq 1$.