Preliminary Examination in Analysis

January 2026

Instructions

- This is a three-hour exam on Advanced Calculus and Real Analysis.
- Please work a total of five problems (four mandatory problems, two from each section, and one optional problem). You *must* work the mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Please indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose that $A, B \subset \mathbb{R}$ are nonempty and bounded above, and let

$$A + B = \{a + b : a \in A, b \in B\}.$$

   Prove that

$$\sup(A + B) = \sup A + \sup B.$$

2. Suppose that f is continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 0$. Show that f is uniformly continuous on $[0, \infty)$.

Advanced Calculus, Optional Problems

3. (a) State the Weierstrass approximation theorem.

   (b) Suppose that $f \in C[0, 1]$ and that

$$\int_0^1 x^n f(x)\, dx = 0$$

   for all $n \geq 0$. Prove that f is the zero function.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $f(0) = f'(0) = 0$. Show that, if $\{a_n\}$ is positive and $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} f(a_n)$ converges.

Real Analysis, Mandatory Problems

1. Suppose $E \subset \mathbb{R}$ has the property that for every open interval I,

$$m_*(E \cap I) \leq \frac{1}{2}|I|.$$

Show that $m_*(E) = 0$. Note that here $m_*(E)$ denotes the outer measure of E. Hint: Recall that every open set in $\mathbb{R}$ can be written as the disjoint union of open intervals.

2. Suppose that f is a Lebesgue-integrable function in $\mathbb{R}^d$ and let

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x)\, dx.$$

Prove that $\widehat{f}$ is a bounded and continuous function on $\mathbb{R}^d$.

Real Analysis, Optional Problems

3. Suppose $\{q_1, q_2, \ldots\} = \mathbb{Q} \cap [0,1]$ is an enumeration of the rational numbers in $[0,1]$, and let

$$E_n = [q_n, q_n + 1/n] \cup [1 + q_n, 1 + q_n + 1/n] \cup [2 + q_n, 2 + q_n + 1/n] \cup \ldots$$

Show that if $f : \mathbb{R} \to \mathbb{R}$ is integrable, then

$$\lim_{n \to \infty} \int_{E_n} f = 0.$$

4. This problem concerns measurable functions $f : E \subset \mathbb{R} \to \mathbb{R}$.

   (a) State Egoroff's theorem.

   (b) Suppose that $E \subset \mathbb{R}$ is a bounded measurable set with $m(E) < \infty$, and that $\{f_n\}_{n=1}^{\infty}$ is a sequence of uniformly bounded, real-valued, measurable functions on E with $f_n \to f$ pointwise for a.e. x. Using Egoroff's theorem, prove that

$$\lim_{n \to \infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx.$$