6/10/p1

ANALYSIS PRELIMINARY EXAM

June 7, 2010

NAME: ______________________________ OPTION: ______________

Instructions

1. This is a three hour examination which consists of two parts: **Advanced Calculus and Real** *or* **Complex Analysis.** You must work problems from the section on Advanced Calculus and from the section of the option, Real or Complex, that you have chosen.

2. You need to work a total of 5 problems: Four mandatory problems (two mandatory problems from each of two parts), and one optional problem from either advanced calculus or the option you have chosen. Please indicate clearly on your test paper whether you are taking the **Real or Complex Analysis option**, and which optional problem you are solving. **Only 5 problems will be graded.**

3. Do not put your name on any sheet except the cover page. The papers will be blind-graded.

4. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to partial solutions of the easy parts of two different problems. Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. (a) Let (X, d) be a metric space and $E \subset X$ a subset. Prove that if $f_n : E \to \mathbb{R}$ is a sequence of continuous functions and $f_n \to f$ uniformly on E, then $f : E \to \mathbb{R}$ is continuous.

   (b) For $(X, d) = (\mathbb{R}, |\cdot|)$, present an example in which $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions and $f_n \to f$ on $[0,1]$, but $f : [0,1] \to \mathbb{R}$ is discontinuous.

2. Suppose that (X, d_1) and (Y, d_2) are two metric spaces, and $f : X \to Y$ is continuous. Show that $f(K) \subset Y$ is a compact set, provided $K \subset X$ is a compact set.

Advanced Calculus, Optional Problems

1. (a) For a bounded function on a bounded closed interval $f : [a, b] \to \mathbb{R}$, state the precise definition that f is *Riemann-integrable* on $[a, b]$.

   (b) Prove that if $f : [a, b] \to \mathbb{R}$ is monotonic, then f is *Riemann-integrable* on $[a, b]$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$|f'(x)| \le \frac{1}{2}, \ \forall x \in \mathbb{R}.$$

Define a sequence

$$x_{n+1} = f(x_n), \ n = 1, 2, \cdots,$$

and $x_1 = 10$. Show that

$$\lim_{n \to \infty} x_n$$

exists and is finite.

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Real Analysis, Mandatory Problems

1. (a) State the Lebesgue Dominated Convergence Theorem.

   (b) Let $\{f_n\}$ be a sequence of Lebesgue integrable functions on $E = [0, 1]$. Use an example to show that $f_n(x)$ converges to $f(x)$ for every $x \in E$ does not imply that $\int_E f_n \to \int_E f$ as $n \to \infty$, even if f is Lebesgue integrable on E.

   (c) Let f be a Lebesgue integrable function on $[0, 1]$. Find

$$\lim_{n\to\infty} \int_0^1 \frac{nx^2}{1+nx} f(x)\, dx.$$

   Justify your answer.

2. Let $\{r_k\}$ denote the rational numbers in $[0, 1]$ and $\{a_k\}$ satisfy $\sum |a_k| < \infty$. Show that $\sum a_k |x - r_k|^{-1/2}$ converges a.e. in $[0, 1]$.

Real Analysis, Optional Problems

1. Let f be a measurable function on $[0, 1]$. Suppose that

$$M = \inf\{\alpha : |\{x \in [0,1] : \ |f(x)| > \alpha\}| = 0\} < \infty.$$

   Show that $|f(x)| \le M$ for a.e. $x \in [0, 1]$ and

$$\lim_{p\to\infty} \left\{ \int_0^1 |f(x)|^p\, dx \right\}^{1/p} = M.$$

2. Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for a.e. $x \in \mathbb{R}^1$, $f(x, y)$ is finite for a.e. $y \in \mathbb{R}^1$. Show that for a.e. $y \in \mathbb{R}^1$, $f(x, y)$ is finite for a.e. $x \in R^1$.

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Complex Analysis, Mandatory Problems

1. Let $n \geq 2$ be a positive integer, and set

$$f(z) = \frac{1}{1+z^n}.$$

Let Γ denote the path consisting of the segment $0 \leq x \leq R$ on the real axis, the arc $z = Re^{it}$, $0 \leq t \leq \frac{2\pi}{n}$, on the circle $|z| = R$, and the segment from $Re^{\frac{2\pi i}{n}}$ to 0 on the line $\arg z = \frac{2\pi}{n}$.

(a) Compute $\int_\Gamma f(z)dz$.

(b) Let $R \to \infty$ and obtain the formula

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n\sin(\frac{\pi}{n})},$$

a formula known to Euler in 1743.

2. (a) State Rouché's theorem on the zeros of analytic functions.

(b) Find the number of zeros of the polynomial $p(z) = 2z^5 + 8z - 1$ lying in the annulus $1 < |z| < 2$.

Complex Analysis, Optional Problems

1. (a) Suppose that f is analytic in a deleted neighborhood $\Omega = \{z : 0 < |z-a| < \epsilon\}$. Prove that if $|f(z)| \leq M$ in Ω, then f has a removable singularity at $z = a$.

(b) Suppose that f and g are entire functions with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant c so that $f = cg$.

2. Let f be an entire function with $|f(z)| \leq 3\log|z|$ in some neighborhood of ∞. Prove that f must be constant.