June 11/p1

Analysis Preliminary Exam

3 June 2011

Instructions

1. This is a three hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis. You must work problems from the Advanced Calculus section and the Real Analysis section or the Complex Analysis section, depending on the option you have chosen.

2. You should attempt a total of five problems: four mandatory problems (two from each section) and one optional problem from either section. Please indicate clearly whether you are taking the Real Analysis option or the Complex Analysis option and which optional problem is to be graded. If you do not indicate which optional problem is to be graded, the one with the lowest score will be used to determine your grade.

3. Do not put your name on any sheet except the cover sheet. The exam will be blind-graded.

4. Each question is weighted equally.

5. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution to one problem than to solutions of the easy bits in two different problems.

6. Give complete and detailed arguments. Indicate clearly the definitions and proofs that you are using.

Due 11/p2

ADVANCED CALCULUS

MANDATORY PROBLEMS

1. Let f be a positive real-valued function defined on the natural numbers and suppose f is decreasing. Show that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\sum_{n=0}^{\infty} 2^n f(2^n)$ converges.

2. A family $\mathcal{F}$ of sets is said to have the *finite intersection property* if every finite intersection of sets of $\mathcal{F}$ is nonempty. Show that a metric space is compact if and only if every family of closed sets in the space with the finite intersection property has a nonempty intersection.

OPTIONAL PROBLEMS

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative continuous functions on $[0,1]$ that satisfy $f_{n+1} \leq f_n$ for $n = 1, 2, \ldots$ and suppose that for each x in $[0,1]$, we have
$$\lim_{n\to\infty} f_n(x) = 0.$$
Show that the sequence $\{f_n\}$ converges uniformly to zero.

4. Given a number $b > 0$, define a sequence $\{a_n\}$ inductively by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{b}{a_n})$. Prove that the sequence $\{a_n\}$ converges.

Jensell/p3

Real Analysis
Mandatory problems

If E is a Lebesgue measurable set, we let $m(E)$ denot the Lebesgue measure of E.

1. (a) Let $E \subset \mathbf{R}$ be a Lebesgue measurable set with finite measure and suppose $f : E \to \mathbf{R}$ is a measurable function. Prove that for given $\epsilon > 0$, we may find a set $F \subset E$ so that $m(E \setminus F) < \epsilon$ and f is bounded on F.

   (b) Give an example to show that the result of part a) may fail if $E = \mathbf{R}$.

2. Let f be a non-negative measurable function on the real line and suppose that

$$\int_{\mathbf{R}} f(t)\, dt$$

is finite. Show that

$$\lim_{\alpha \to \infty} \alpha\, m(\{t : f(t) > \alpha\}) = 0.$$

Optional problems

3. Let $\{r_1, r_2, \ldots\}$ be an enumeration of the rationals in the unit interval $[0,1]$ and consider the sum

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} |x - r_k|^{-1/2}.$$

Find the measure of the set $\{x \in [0,1] : f(x) < \infty\}$.

4. (a) Give the definition of the Hardy-Littlewood maximal function, f^*.

   (b) Let f be in $L^1(\mathbf{R})$ and define

$$P(x) = \int_{\mathbf{R}} f(x-y) \frac{1}{1+|y|^2}\, dy.$$

Find a finite constant C so that

$$|P(x)| \le C f^*(x).$$

June 11/p4

Complex Analysis
Mandatory problems

1. Suppose that f is a non-constant analytic function in $\{z : |z| < R\}$.

   (a) Let
   $$M(r) = \max\{|f(z)| : |z| = r\}, \qquad 0 < r < R$$
   and show that $M(r)$ is monotone increasing.

   (b) Show that if this f is actually entire and not a polynomial, then for every integer $n \geq 0$,
   $$\lim_{r\to\infty} \frac{M(r)}{r^n} = +\infty.$$

2. Let $f(z)$ be analytic in the set $\{z : |z| < 2, z \neq 1\}$ and f has a simple pole at $z = 1$ with residue b. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\{z : |z| < 1\}$, show that
   $$\lim_{n\to\infty} a_n = -b.$$
   Hint: Consider the function $g(z) = f(z) - \frac{b}{z-1}$.

Optional problems

3. Show that all five of the zeroes of the polynomial
   $$f(z) = z^5 + 15z + 1$$
   lie in the disk $\{z : |z| < 2\}$ and that only one zero lies inside the disk $\{z : |z| < 3/2\}$.

4. Find the Laurent series expansion in the annulus $\{z : 1 < |z| < 4\}$ for the function
   $$f(z) = \frac{z+6}{(z+1)(z-4)}.$$