Analysis Preliminary Exam

6 June 2012

Instructions

1. This is a three hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis. You must work problems from the Advanced Calculus section and the Real Analysis section or the Complex Analysis section, depending on the option you have chosen.

2. You should attempt a total of five problems: four mandatory problems (two from each section) and one optional problem from either section. Please indicate clearly whether you are taking the Real Analysis option or the Complex Analysis option and which optional problem is to be graded. If you do not indicate which optional problem is to be graded, the one with the lowest score will be used to determine your grade.

3. Do not put your name on any sheet except the cover sheet. The exam will be blind-graded.

4. Each question is weighted equally.

5. You should provide complete and detailed solutions to each problem that you work. More weight will be given for a complete solution to one problem than for solutions of the easy bits in two different problems.

6. Indicate clearly the definitions and theorems that you are using.

Advanced calculus
Mandatory problems

1. (a) Let $A \subset \mathrm{R}$ be a bounded set and suppose that f is uniformly continuous on A. Show that f is a bounded function.

   (b) If $A \subset \mathrm{R}$ is a bounded set and f is a continuous function on A, must f be bounded?

2. Suppose that f is continuous and non-negative on $[0,1]$ and $\int_0^1 f(t)\,dt = 0$. Prove that $f = 0$.

Optional problems

3. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence and suppose that $\lim_{n\to\infty} a_n = 1$. Let

$$s_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

   Find the limit $\lim_{n\to\infty} s_n$.

4. Let $\lfloor x \rfloor$ denote the integer part of x, $\lfloor x \rfloor = \max(\mathrm{Z} \cap (-\infty, x])$ with Z denoting the integers.

   Determine if the limit

$$\lim_{N\to\infty} \int_1^N \frac{1}{\lfloor x \rfloor} - \frac{1}{x}\,dx$$

   exists. If the limit exists, determine if it is finite.

Real Analysis, Mandatory Problems

1. This problem concerns Lebesgue measure on $\mathbb{R}^d$.
(a) Define what it means for a subset E of $\mathbb{R}^d$ to be measurable.
(b) Suppose that A and B are measurable sets of finite measure with $m(A) = m(B)$, and suppose that $A \subset E \subset B$. Show that E is a measurable set.

2. This problem concerns measurable functions $f : E \subset \mathbb{R} \to \mathbb{R}$.
(a) State Egoroff's Theorem.
(b) Suppose that $E \subset \mathbb{R}$ is a measurable set with $m(E) < \infty$, and that $\{f_n\}_{n=1}^{\infty}$ is a sequence of uniformly bounded, real-valued, measurable functions on E with $f_n \to f$ pointwise for a.e. x. Using Egoroff's Theorem, prove that

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx.$$

Real Analysis, Optional Problems

3. Suppose that f is a nonnegative integrable function on $\mathbb{R}^d$. Prove that for any $\varepsilon > 0$, there is a $\delta > 0$ so that

$$\int_E f(x)\,dx < \varepsilon, \text{ whenever } m(E) < \delta.$$

4. Suppose that f is a continuous real-valued function and consider the curve $\Gamma \subset \mathbb{R}^2$ given by $\{(x, f(x)) : x \in \mathbb{R}^d\}$. Prove that Γ has measure zero as a subset of $\mathbb{R}^2$.