Preliminary Examination in Analysis

June 2020

Instructions

- This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.
- You should work problems from the section on advanced calculus and from the section of the option you have chosen.
- You are to work a total of five problems (four mandatory problems and one optional problem).
- You must work two mandatory problems from each part.
- Please indicate clearly on your test paper which optional problem is to be graded.
- Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Suppose that $A \subset \mathbb{R}$ is a nonempty set with the property that if $r \in A$, then $\sqrt{r^2+1} \in A$. Show that A is not bounded above.

2. Let $\{f_n\}$ be a sequence of differentiable functions on $[0,1]$. Suppose that f_n converges pointwise to zero on $[0,1]$, and that $|f_n'(x)| \leq 1$ for all $n \geq 1$ and all $x \in (0,1)$. Show that f_n converges uniformly to zero on $[0,1]$.

Advanced Calculus, Optional Problems

3. Suppose $f : [0,1] \to [0,1]$ has the property that for each $n \geq 1$, there exist exactly n points at which $f(x) \geq (1/n)$. Show that f is Riemann integrable on $[0,1]$.

4. Suppose that $A \subset \mathbb{R}$ contains a sequence $\{a_n\}$ such that $a_n \neq 0$, but $\lim_{n\to\infty} a_n = 0$. Also assume that A has the property that if $a, b \in A$, then $a+b, a-b \in A$. Show that A is dense in $\mathbb{R}$.

Real Analysis, Mandatory Problems

1. Let f be an integrable function in $\mathbb{R}^d$. Construct a sequence $\{f_n\}$ of bounded measurable functions with compact support such that

$$\int_{\mathbb{R}^d} |f_n - f| \, dx \to 0 \quad \text{ as } n \to \infty.$$

You need to prove the convergence.

2. Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}^d$ with

$$m(E_k) \le \frac{1}{k^2} \quad \text{ for any } k \ge 1.$$

Let

$$E = \big\{ x \in \mathbb{R}^d : \; x \in E_k \text{ for infinitely many } k \big\}.$$

(a) Show that E is measurable.

(b) Show that $m(E) = 0$.

Real Analysis, Optional Problems

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f(E)$ is a Borel set if E is a Borel set.

4. Let f be an integrable function in $(0, 1)$. Suppose that

$$\int_0^1 fg \, dx \ge 0$$

for any nonnegative continuous function $g : (0, 1) \to \mathbb{R}$. Prove that $f \ge 0$ a.e. in $(0, 1)$.

Complex Analysis, Mandatory Problems

1. Compute the following integral

$$\int_0^\infty \frac{\cos(\pi x)}{(x^2+1)^2}\, dx.$$

Hint: Use the first few terms of the Laurent expansion to calculate the residue(s). Justify all steps of the calculation.

2. Suppose g is an entire function and its real part $u = \Re(g)$ satisfies the following : There exist a constant $s \geq 0$ and a constant $0 \leq C < \infty$ so that for all $n \geq 1$,

$$|u(x, y)| \leq C n^s, \qquad \text{for } |z| = n,$$

where $z = x + iy$. Prove that g is a polynomial of degree less than or equal to s. Hint: Express the coefficients of the Taylor expansion as integrals over circles of increasing radii. Show that the related integrals corresponding to negative indices vanish and combine these to get information depending on u.

Complex Analysis, Optional Problems

3. Suppose that $\{f_n\}$ is a sequence of analytic functions on the unit disk

$$D = \big\{ z \in \mathbb{C} : |z| < 1 \big\}$$

such that $f_n \to f$ uniformly on the set $\{z \in D : |z| = r\}$ for each $0 < r < 1$.

(a) Prove that $\{f_n\}$ converges uniformly on any compact subset of D to f and that f is analytic on D.

(b) Prove that the sequence $\{f_n'\}$ converges uniformly on any compact subset of D to f'.

4. Let f be analytic on the unit disk D and continuous on its closure $\overline{D}$. Suppose that $|f(z)| = 1$ for all $|z| = 1$.

(a) Prove that if f never vanishes in D, then f is a constant.

(b) Prove that there are only finitely many zeros of f in D.

(c) Suppose $a_1, a_2, \ldots, a_n$ are the zeros of f in D. Prove that there is an angle $\theta \in \mathbb{R}$ so that

$$f(z) = e^{i\theta} \left(\frac{z - a_1}{1 - \bar{a}_1 z} \right) \cdots \left(\frac{z - a_n}{1 - \bar{a}_n z} \right).$$