Preliminary Examination in Analysis

June 7, 2021

Instructions

• This is a three-hour examination which consists of two parts: Advanced Calculus and Real or Complex Analysis.

• You should work problems from the section on advanced calculus and from the section of the option that you have chosen.

• You are to work a total of five problems (four mandatory problems and one optional problem).

• You must work the two mandatory problems from each part.

• Please indicate clearly on your test paper which optional problem is to be graded.

• Indicate clearly what theorems and definitions you are using.

Advanced Calculus, Mandatory Problems

1. Let f be a real-valued continuous function in $[0, \infty)$. Suppose that

$$\lim_{x\to\infty} f(x) = 0.$$

Show that f is uniformly continuous on $[0, \infty)$.

2. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions.
 i) What does it mean for f_n to converge uniformly to $f : [0, 1] \to \mathbb{R}$?
 ii) Assume that f_n converges to f uniformly. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

Advanced Calculus, Optional Problems

3. Assume that A and B are nonempty subsets of $(0, \infty)$. Define

$$AB = \{ab : a \in A, b \in B\}$$

Show that

$$\sup(AB) = (\sup A)(\sup B)$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function so that $f(1) = 42$ and

$$|f(x) - f(y)| \leq 100|x - y|^2$$

for all $x, y \in \mathbb{R}$. Show that $f(x) = 42$ for all $x \in \mathbb{R}$.

Real Analysis, Mandatory Problems

1. Assume that $f : [0,1] \to \mathbb{R}$ is a continuous function. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = f(x^n)$. Show that f_n is integrable, and compute $\lim_{n\to\infty} \int_{[0,1]} f_n(x)dx$

2. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz function, i.e. there is $M > 0$ so that $|f(x)-f(y)| \leq M|x-y|$ for all $x, y \in \mathbb{R}^d$.

i) Find a constant $\widetilde{M} > 0$ so that, for any cube Q, we have that

$$m_*(f(Q)) \leq \widetilde{M}|Q|,$$

where m_* is the exterior measure.

ii) Show that for any $E \subset \mathbb{R}^d$ with $m(E) = 0$ we have that $m(f(E)) = 0$, where m is the Lebesgue measure.

Real Analysis, Optional Problems

3. Let $f(x, y)$ be a nonnegative and measurable function in $\mathbb{R}^2$. Suppose that for a.e. $x \in \mathbb{R}$, $f(x, y)$ is finite for a.e. $y \in \mathbb{R}$. Show that for a.e. $y \in \mathbb{R}$, $f(x, y)$ is finite for a.e. $x \in \mathbb{R}$.

4. Let $\{f_k\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$. Suppose that $f_k \to f$ and $f_k \leq f$ a.e. in $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} f_k \, dx \to \int_{\mathbb{R}} f \, dx.$$

Complex Analysis, Mandatory Problems

1. Prove the following version of Hurwitz's Theorem:

Theorem: *Suppose that $f_n : \mathcal{A} \subset \mathbb{C}$ is a sequence of functions analytic and non-vanishing on an open, connected subset $\mathcal{A}$. Suppose that $f_n \to f$ uniformly on any compact subset of $\mathcal{A}$. Then the limit functions f is either identically zero on $\mathcal{A}$ or never zero on $\mathcal{A}$.*

2. Evaluate the following integral:

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}\, dx.$$

Make sure that you carefully describe all the steps.

Complex Analysis, Optional Problems

3. This problem concerns conformal maps.

a: Prove that there is no fractional linear transformation T satisfying $Tz = \overline{z}$ for all $z \in \mathbb{C}$.

b: Prove that the map $Tz = \overline{z}$, for $z \in \mathbb{D}$, the unit disk, is a conformal map of $\mathbb{D}$ to itself. Characterize T as an element of the automorphism group of $\mathbb{D}$. (Recall the general form of any $S \in \mathrm{Aut}(\mathbb{D})$.)

c: What is the automorphism $\widetilde{T}$ of the upper-half complex plane $\mathbb{H}$ corresponding to the transformation in part (b). Describe the effect of $\widetilde{T}$ geometrically. Recall that the map $S : \mathbb{H} \to \mathbb{D}$ is given by $Sz = (z-i)(z+i)^{-1}$.

4. Let $\Omega \subset \mathbb{C}$ be a bounded region symmetric with respect to the real line $\mathbb{R}$: If $z \in \Omega$, then $\overline{z} \in \Omega$. Let $\Omega^\pm := \{z \in \Omega \mid \Im z > 0 \text{ or } \Im z < 0\}$, and let $\Sigma := \Omega \cap \mathbb{R}$. We suppose that $\Sigma = (a,b)$, for two finite real numbers $a < b$.

a: Suppose $f : \Omega^+ \to \mathbb{C}$ is analytic on Ω^+ and continuous on $\Omega^+ \cup \Sigma$, and real on Σ. Construct a function $\widetilde{f}$ on Ω that is analytic on $\Omega^+ \cup \Omega^-$, continuous on Ω and real on Σ.

b: Prove that $\widetilde{f}$ is analytic on Ω by applying Morera's Theorem on triangular contours in small disks centered at points $x_0 \in \Sigma$. Conclude that f admits an analytic extension $\widetilde{f} : \Omega \to \mathbb{C}$.