# Preliminary Examination in Analysis

**June 2026**

## Instructions

- This is a three-hour exam on Advanced Calculus and Real or Complex Analysis.
- Please work a total of five problems (four mandatory problems, two from each section, and one optional problem). You *must* work the mandatory problems from each part
- Please indicate clearly on your test paper which optional problem is to be graded
- Please indicate clearly what theorems and definitions you are using

# Advanced Calculus, Mandatory Problems

1. Assume that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $f \geq 0$. The goal is to prove that $f^2$ is Riemann integrable.

   i) Prove that for a bounded set $A \subset [0, \infty)$, we have $\sup(A^2) = (\sup A)^2$. Prove the corresponding result for the infimum.

   ii) Apply the result of part (i) to prove that $f^2$ is Riemann integrable on $[a, b]$.

2. (i) Prove that if $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ that converges uniformly on $[a, b]$ to a function $f$, then $f$ is also continuous on $[a, b]$.

   (ii) Give an example of a sequence of continuous functions $\{f_n\}$ on $[a, b]$ that converges pointwise to a function $f$ that is not continuous.

# Advanced Calculus, Optional Problems

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$. If $\lim_{x \to 0} f'(x) = 0$, show that $f'(0)$ exists, and that $f'(0) = 0$.

4. Suppose that $\{f_n\}$ is a decreasing sequence of nonnegative, continuous functions on $[0, 1]$:

   $$f_1(x) \geq f_2(x) \geq \ldots \geq f_n(x) \geq f_{n+1}(x) \geq \ldots \geq 0.$$

   In addition, suppose that $\lim_{n \to 0} f_n(x) = 0$ for each $x \in [0, 1]$. Prove that $f_n \to 0$ uniformly on $[0, 1]$, and that $\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = 0$.

## Real Analysis, Mandatory Problems

1. Let $E \subset \mathbb{R}^d$ be any set, and define $2E = \{2x \mid x \in E\}$.

   (i) Prove that $m_\star(2E) = 2^d m_\star(E)$, where $m_*$ denotes the exterior (or outer) measure on $\mathbb{R}^d$.

   (ii) Use part (i) to prove that, if $E$ is Lebesgue measurable, then so is $2E$, and compute $m(2E)$.

2. (i) State and prove the Lebesgue Dominated Convergence Theorem using the Bounded Convergence Theorem.

   (ii) Let $f \in L^1(\mathbb{R}^d)$ and define the function $K_n(x)$ by

   $$K_n(x) := e^{-\frac{|x|^2}{n^2}}.$$

   Compute the limit

   $$\lim_{n\to\infty} \int_{\mathbb{R}^d} K_n(x) f(x)\, dx.$$

   Make sure to justify your answer.

## Real Analysis, Optional Problems

3. Let $E \subset \mathbb{R}^d$ be measurable. Let $f \in L^1(E)$ be a nonnegative function $f(x) \geq 0$ a. e. $x \in E$. Define $E_\alpha := \{x \in E \mid f(x) \geq \alpha > 0, \quad x \in E\}$. Prove that

   $$\int_E f = \int_0^\infty m(E_\alpha)\, d\alpha.$$

   HINT: Fubini's Theorem might be helpful.

4. (i) Let $F, G : [a, b] \to \mathbb{R}$ be absolutely continuous functions. Show that $FG$ is absolutely continuous.

   (ii) Prove that $F$ and $G$ satisfy the integration-by-parts identity:

   $$\int_a^b F'(x)G(x)\, dx = -\int_a^b F(x)G'(x)\, dx + [F(b)G(b) - F(a)G(a)].$$

   Be sure to justify your computations.

# Complex Analysis, Mandatory Problems

1. Compute the integral
$$\int_0^\infty \frac{e^{ax}}{1+e^x}dx$$
for $0 < a < 1$, using the rectangular contour with vertices at $-R, R, R+2\pi i, -R+2\pi i$. Justify all the steps of the calculation.

2. Let
$$\mathbb{D}_1 := \{z \in \mathbb{C} : |z| < 1\}, \;\; \mathbb{D}_2 := \{z \in \mathbb{C} : |z| < 2\}.$$
Let $f : \mathbb{D}_1 \to \mathbb{D}_2$ be a holomorphic function with $f(0) = 0$.
   (a). Prove that $|f(z)| \le 2|z|$ for all $z \in \mathbb{D}_1$.
   (b). Prove that if $f$ is bijective then $|f(z)| > |z|$ for all $z \in \mathbb{D}_1 \setminus \{0\}$.

# Complex Analysis, Optional Problems

1. Let $f$ be a holomorphic function on a region containing the closed unit disk $\{z \in \mathbb{C} : |z| \le 1\}$. Suppose that
   (i). $f(0) = 0$;
   (ii). $f(iz) = f(z)$ for all $|z| < 1$;
   (iii). $|f(z)| \le 2026$ for all $|z| < 1$.

   Prove that

   (a). $f$ has a zero of order at least $4$ at the origin; (Hint: Consider the power series expansion of $f$ at $0$)
   (b). $|f(0.1)| < 1$.

2. Let $f$ be an entire function such that
$$|f(z)| \le C|z|^{1/2}$$
for all $z \in \mathbb{C}$ and some constant $C$. Prove that $f$ is constant.