Preliminary Examination in Numerical Analysis

January 9, 2009

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.
5. Problems within each problem set are not necessarily related but they may be. You could use the result from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problem sets in this part)

Problem 1.

a) Let A be an $n \times n$ nonsingular matrix and let $\hat{x}$ be an approximate solution to $Ax = b$. If $r = A\hat{x} - b$, prove that
$$(A + E)\hat{x} = b$$
where E is a matrix satisfying $\|E\|_2 = \frac{\|r\|_2}{\|\hat{x}\|_2}$.

b) Let U be an $n \times n$ upper triangular matrix and consider solving $Ux = b$ by backward substitution in a floating point arithmetic. Prove that the computed solution $\hat{x}$ satisfies $(U + \delta U)\hat{x} = b$ with $|\delta U| \le n\epsilon |U| + O(\epsilon^2)$, where ϵ is the machine precision.
(You may use $fl(\sum_{i=1}^{d} x_i y_i) = \sum_{i=1}^{d} x_i y_i (1 + \delta_i)$ with $|\delta_i| \le d\epsilon + O(\epsilon^2)$.)

c) Write down the algorithm for computing the Cholesky factorization of a symmetric positive definite matrix.

Problem 2.

a) Describe an efficient algorithm to reduce the following matrix
$$A = \begin{pmatrix} \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ & & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ & & & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ & & & & \mathrm{x} & \mathrm{x} \end{pmatrix}$$
to the upper Hessenberg form by an orthogonal similarity transformation. (A sketch of a procedure is sufficient.)

b) State the power method for computing the largest eigenvalue (in absolute value) of a matrix A. Assuming that A is diagonalizable, i.e. $A = P\Lambda P^{-1}$ with Λ being diagonal, state and prove the convergence property of the power method.

c) For $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ and $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, find the sequence generated by the power method for A with x_0 as the initial vector. Discuss the convergence property of the sequence obtained. Is there any contradiction to the result in (b)? Explain.

Problem 3. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \geq n)$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \in R^{m\times n}; \;\; \Sigma_1 := \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix}$$

with $\sigma_1 \geq \cdots \geq \sigma_k > 0$ is $k \times k$.

a) Determine $\mathcal{R}(A)$ and $\mathcal{N}(A)$ in terms of the left and right singular vectors, where $\mathcal{R}(A)$ denotes the range space of A and $\mathcal{N}(A)$ denotes the null space of A.

b) Prove that $P = AA^\dagger$ is an orthogonal projection, i.e. P is symmetric and $P^2 = P$.

c) Prove that $\mathcal{R}(P) = \mathcal{R}(A)$ and $\mathcal{R}(I - A^\dagger A) = \mathcal{N}(A)$.

Part II – Numerical Analysis
(Work two of the three problems in this part)

Problem 4.

(a) Let $x \in [0, \pi]$ and $0 < \epsilon < 1$. Consider the problem of finding $y \in \mathbb{R}$ such that $x = y - \epsilon \sin y$. Interpret this problem as a fixed-point problem, that is, write down a functional iteration (including its domain of definition) that will converge to such a point y.

(b) Prove that the above iteration converges.

(c) Let $f(x) : \mathbb{R} \to \mathbb{R}$. Write down a theorem that yields the generally best possible order of convergence for Newton's method for finding a root q of f. Be sure to state precisely the conditions under which this theorem is true.

(d) It is easy to check that $f(x) = x^4 + 4x^3 - 3x^2 - 10x + 8$ has a root at $x = 1$. Using Newton's method to approximate this root with $x_0 = 2$ generates the following table.

i	0	1	2	3	4	5	6	7	8	9
x_i	2	1.5862	1.3278	1.1760	1.0918	1.0470	1.0238	1.0120	1.0060	1.0030
$e_i = x_i - 1$	1	0.5862	0.3278	0.1760	0.0918	0.0470	0.0238	0.0120	0.0060	0.0030

Does Newton's method here yield the order of convergence predicted by the theorem stated in part (c) above? If so, verify the conditions of the theorem (at least for x_i close enough to 1). If not, explain why not.

Problem 5.

For parts **(a)** and **(b)**, consider the following theorem on polynomial interpolation:
Theorem. Let $x_0, x_1, ..., x_n$ be $n+1$ distinct real numbers. Then for arbitrary values $y_0, y_1, ..., y_n$, there is a unique polynomial p_n of degree at most n such that $p_n(x_i) = y_i$ for $0 \leq i \leq n$.

(a) Prove the uniqueness assertion of the above theorem.

(b) Prove the existence assertion of the above theorem. *Hint:* It may be easiest to prove existence constructively, i.e., by writing down a formula for p_n.

(c) Determine values of A, B, and C such that

$$\int_0^1 f(x)\mathrm{d}x \approx Af(0) + Bf(\frac{1}{4}) + Cf(1)$$

is exact for all polynomials of degree as high as possible. What is the maximum degree for which the above formula is exact?

(d) Write down a bound for

$$|\int_0^1 \sin(\pi x)\mathrm{d}x - \sum_{i=0}^{n-1} A_i \sin(\pi x_i)|,$$

where $\int_a^b f(x)\mathrm{d}x \approx \sum_{i=0}^{n-1} A_i f(x_i)$ is the Gaussian quadrature rule with n quadrature points.

Problem 6. For parts **(a)**, **(b)**, and **(c)**, consider the linear multistep formula

$$x_n = -4x_{n-1} + 5x_{n-2} + h(4f_{n-1} + 2f_{n-2}).$$

(a) Is the above multistep formula implicit or explicit?

(b) Show that the order of the above method is 3.

(c) Is the above method stable or unstable?

(d) Prove that any consistent 1-step linear multistep formula is stable.