Preliminary Examination in Numerical Analysis

Jan. 5, 2012

Instructions:

1. The examination is for 3 hours.
2. The examination consists of two parts:
   Part I: Matrix Theory and Numerical Linear Algebra
   Part II: Introductory Numerical Analysis
3. There are three problem sets in each part. **Work two out of the three problem sets for each part.**
4. All problem sets carry equal weights.
5. Problems within each problem set are not necessarily related but they may be. You could use the result from one part in your solutions for other parts, even if you did not prove it.

PART I - Matrix Theory and Numerical Linear Algebra
(Work two of the three problem sets in this part)

Problem 1. Let $\mathrm{fl}\,(x)$ denote computational result of an expression x in a floating point arithmetic and let ϵ be the machine roundoff unit.

1. Let A and Q be two $n \times n$ matrices, and assume that Q is orthogonal. Prove that

$$\mathrm{fl}\,(AQ) = (A + E)Q, \quad \|E\|_2 \leq n^3\epsilon\|A\|_2 + \mathcal{O}(\epsilon^2).$$

   (You may use without proof that $\|A\|_2 \leq \sqrt{n}\|A\|_1$ and $\|A\|_1 \leq \sqrt{n}\|A\|_2$.)

2. Let $L = [l_{ij}]$ be an $n \times n$ lower triangular matrix with the diagonals equal to 1. Write down the forward substitution algorithm for solving $Lx = b$. Prove that the computed solution $\hat{x}$ satisfies $(L + \delta L)\hat{x} = b$ with $|\delta L| \leq (n-1)\epsilon|L| + \mathcal{O}(\epsilon^2)$.

Problem 2.

1. Show that an orthogonal matrix that is also upper triangular must be diagonal. What can be said about the diagonal elements?

2. Assume that A and $A + \delta A$ are invertible and $\|A^{-1}\|\|\delta A\| < 1/2$. Prove

$$\frac{\|(A - \delta A)^{-1} - A^{-1}\,(A + \delta A)\,A^{-1}\|}{\|A^{-1}\|} \leq 2\|A^{-1}\|^2\|\delta A\|^2.$$

   where $\|\cdot\|$ is a matrix operator (or subordinate matrix) norm.

3. Let $A \in R^{m\times n}$ and $b \in R^m$ $(m \geq n)$. Prove that x is a solution to the least squares problem $\min_x \|Ax - b\|_2$ if and only if x and some vector r satisfy

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Problem 3.

1. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ be the singular values of $A \in R^{m\times n}$. Prove that

$$\sigma_1 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad \text{and} \quad \sigma_n = \min_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

2. Let $A \in R^{n\times n}$ and let P be a projection such that $PA = AP$. Prove that $\mathcal{R}(P)$ (the range space of P) is an invariant subspace of A.

3. Write down the (single) shifted QR algorithm for a matrix A. Prove that the matrices produced are all similar to the original matrix.

PART 2 - Numerical Analysis

(Work two of the three problem sets in this part)

Problem 4.

1. Let F satisfy $|F'(x)| \leq \lambda < 1$ on the interval $[x_0 - \rho, x_0 + \rho]$, where $\rho = |F(x_0) - x_0|/(1-\lambda)$. Show that the sequence generated by the iteration $x_{n+1} = F(x_n)$ will converge to $x \in [x_0 - \rho, x_0 + \rho]$.

2. Let $f(x) = x \sin x$. Using Newton's method with initial point $x_0 = 1.0$ and double-precision arithmetic generates the following convergence history:

i	0	1	2	3	4	5	6	7
x_i	1.000	0.3910	0.1903	0.09459	0.04722	0.02360	0.01180	0.005900

Now let $f(x) = x \cos x$. Using Newton's method with initial point $x_0 = 0.5$ and double-precision arithmetic generates the following convergence history:

i	0	1	2	3	4
x_i	0.5000	-0.1879	6.962×10^{-3}	-3.375×10^{-7}	3.843×10^{-20}

First, state the orders of convergence observed in both of the above situations. Next explain whether these orders of convergence are the ones expected when Newton's method is used in a "generic" situation, and explain what properties of f lead to any discrepancies from the standard order of convergence.

3. Let $f(x) = \sin(2x)$ and $0 \leq x_0 < x_1 < x_2 < \ldots < x_{12} \leq x_{13} \leq 1$. Let $p(x)$ be the unique polynomial of degree at most 27 satisfying $p(x_i) = f(x_i)$ and $p'(x_i) = f'(x_i)$, $0 \leq i \leq 13$. Find a good upper bound for $\max_{x \in [0,1]} |f(x) - p(x)|$.

Problem 5.

1. Find a Gaussian quadrature formula of the form $\int_0^1 x f(x) dx \approx \sum_{i=0}^{n} A_i f(x_i)$ that is exact for all polynomials of degree 3 or less. (Note carefully the weight $w(x) = x$ in the integral!)

2. Assuming that $x(t)$ solves the initial value problem

$$\begin{cases} x' = -tx^2, \\ x(1) = 2, \end{cases}$$

approximate $x(1.1)$ using one step of a Taylor series method of order two.

Problem 6.

1. Determine a, b, and c so that the function $f(x)$ given below is a quadratic spline:

$$f(x) = \begin{cases} ax, \ x \in (-\infty, 1], \\ b(2-x)^2 + \frac{3}{2}, \ x \in [1, 2], \\ c, \ x \in [2, \infty). \end{cases}$$

Is the function $f(x)$ that you determined also a cubic spline?

2. Assume that $f(x)$ is a smooth function satisfying $f(r) = f''(r) = 0 \neq f'(r)$. Prove that Newton's method converges *cubically* if the initial point x_0 is sufficiently close to r.

3. Use the method of undetermined coefficients in order to generate a linear multistep method of the form

$$x_n = x_{n-1} + h[Af_n + Bf_{n-1} + Cf_{n-2}].$$

Find the order of your method and then determine whether it is convergent.