Preliminary Examination in Numerical Analysis

Jan. 8, 2016

Instructions:

1. The examination is for 3 hours.
2. The examination consists of ten equally-weighted problems. The first five cover Matrix Theory and Numerical Linear Algebra and the last five cover Introductory Numerical Analysis
3. You may **omit one** problem (i.e. work nine out of the ten problems).

Problem 1. Show that if A is an $n \times n$ matrix of real numbers and if b is an $n \times 1$ column vector, then

$$\left\| A \left(I - \frac{bb^T}{b^T b} \right) \right\|_F^2 = \|A\|_F^2 - \frac{b^T A^T A b}{b^T b}.$$

Problem 2. Assume that A and $A + \delta A$ are invertible and $\|A^{-1}\| \|\delta A\| < 1/2$. Prove

$$\frac{\|(A - \delta A)^{-1} - A^{-1}(A + \delta A) A^{-1}\|}{\|A^{-1}\|} \leq 2 \|A^{-1}\|^2 \|\delta A\|^2.$$

where $\|\cdot\|$ is a matrix operator norm.

Problem 3. Let $A \in R^{m \times n}$ and $b \in R^m$ $(m \geq n)$. Consider the least squares problem

$$\min_{x \in R^n} \|Ax - b\|_2. \tag{1}$$

Use the SVD of A to prove that $x = A^\dagger b$ is the solution to (1) with the minimal norm. Describe advantages and disadvantages of the SVD based method for solving (1).

Problem 4. Let μ be an approximate eigenvalue of an $n \times n$ matrix A and x be an approximate eigenvector with $\|x\|_2 = 1$. Let $r = Ax - \mu x$. Show that there is an $n \times n$ matrix E with $\|E\|_2 = \|r\|_2$ such that $A + E$ has eigenvalue μ with a corresponding eigenvector x.

Problem 5. Write down the (single) shifted QR algorithm for a matrix A. Prove that the matrices produced are all similar to the original matrix. Explain how and why the shifting may accelerate convergence.

Problem 6. Let w be a continuous positive function on the open interval (a, b) and let N be a positive integer. Suppose $p_0, \ldots, p_N$ are real polynomials of a single variable such that p_i has degree i and $(p_i, p_j) = \delta_{i,j}$ whenever $0 \leq i, j \leq N$. Here

$$(p, q) = \int_a^b p(x) q(x) w(x)\, dx$$

for real polynomials p and q of a single variable and $\delta_{i,j} = 1$ when $i = j$ and $\delta_{i,j} = 0$ otherwise. Show that there exist constants A_i, B_i and C_i such that

$$p_{i+1}(x) = (A_i x + B_i) p_i(x) - C_i p_{i-1}(x)$$

for all i satisfying $0 \leq i < N$, where we take $p_{-1} = 0$. (Hint: Choose A_i so that $p_{i+1}(x) - A_i x p_i(x)$ is a polynomial of degree at most n and write this polynomial as a linear combination of $\{p_i : i \leq n\}$.)

Problem 7. Suppose that p interpolates $f \in C^{n+2}[a, b]$ at $a \leq x_0 < \ldots < x_n \leq b$. Show that there exists $\xi_i \in (a, b)$ such that

$$p'(x_i) - f'(x_i) = \frac{f^{(n+1)}(\xi_i)}{(n+1)!} \prod_{\substack{0 \leq j \leq n \\ j \neq i}} (x_i - x_j).$$

Problem 8. Suppose that a function $f \in C^1[a,b]$ has a simple root at $c \in (a,b)$. Consider perturbing the function to $f + \varepsilon g$ where $g \in C^1[a,b]$. Show that as $\varepsilon \to 0$ the condition number of the root at c is $1/|c||f'(c)|$.

Problem 9. Suppose you have a quadrature rule of the form $I_n(f) = \sum_{i=1}^{n} w_i f(x_i)$ that integrates polynomials up to degree n on the interval $[a,b]$ exactly. Show that for $f \in C^{n+1}[a,b]$ the error in the rule can be written in the form

$$I(f) - I_n(f) = \frac{1}{n!} \int_a^b f^{(n+1)}(t) K(t) dt,$$

where $K(t) = I((x-t)_+^n) - I_n((x-t)_+^n)$ depends only on the quadrature rule and is independent of f. The function $(x-t)_+$ is zero for $x \leq t$ and equal to $x - t$ for $x > t$. What must be true of $K(t)$ if the quadrature rule integrates polynomials of degree $n+1$ exactly. (Hint: Use the integral form of the remainder in Taylor's theorem.)

Problem 10. Let $f : \mathbb{R} \to \mathbb{R}$ have a continuous derivative and suppose f has a fixed point $p \in \mathbb{R}$ with $|f'(p)| < 1$. Show that there is a $\delta > 0$ such that if $|x_0 - p| < \delta$ and if $\{x_n\}_{n=0}^{\infty}$ is the sequence of iterates with $x_{n+1} = g(x_n)$ for $n \geq 0$, then $\{x_n\}_{n=0}^{\infty}$ converges to p.