# Preliminary Examination in Numerical Analysis

Jan 2, 2019

**Instructions:**

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

**Problem 1**. Let $\mathrm{fl}\,(x)$ denote computational result of an expression $x$ in a floating point arithmetic and let $\epsilon$ be the machine roundoff unit.

(a) Show that

$$\mathrm{fl}\,(\sum_{i=1}^{n} x_i y_i) = \sum_{i=1}^{n} x_i y_i (1+\delta_i)$$

with $\delta_i \leq n\epsilon + \mathcal{O}(\epsilon^2)$.

(b) Let $A$ and $X$ be two $n \times n$ matrices, and assume that $X$ is nonsingular. Show that

$$\mathrm{fl}\,(AX) = (A+E)X, \quad \|E\|_1 \leq (n\epsilon + \mathcal{O}(\epsilon^2))\kappa_1(X)\|A\|_1,$$

where $\kappa_1(X) = \|X\|_1 \|X^{-1}\|_1$.

**Problem 2**. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Consider the regularized least-squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|_2^2 + \mu^2\|x\|_2^2 \tag{1}$$

(a) Show that $\hat{x} = (A^TA + \mu^2 I)^{-1}A^Tb$ is the solution to (1).

(b) Assume that $A$ has full column rank. Show that $\kappa_2(A^TA+\mu^2 I) \leq \kappa_2(A^TA)$, where $\kappa_2(M) = \frac{\lambda_{\max}(M)}{\lambda_{\min}(M)}$ denotes the spectral condition number of a symmetric positive definite matrix $M$.

**Problem 3**. Let $A$ be a symmetric $n \times n$ matrix and $\lambda_1$ its largest eigenvalue. Show that

$$\max_{x\neq 0} \frac{x^TAx}{x^Tx} = \lambda_1 \tag{2}$$

and the maximum is attained at any eigenvector corresponding to $\lambda_1$.

**Problem 4**. Let $A$ be an $n \times n$ symmetric positive definite matrix.

(a) Show that there is a symmetric positive definite matrix $S$ such that $A = S^2$.

(b) For any $x, y \in \mathbb{R}^n$, show that

$$(x^Ty)^2 \leq (x^TAx)(y^TA^{-1}y)$$

**Problem 5**. Assume that $f(x)$ is a smooth function and $r$ is a root of multiplicity $m$, i.e. $f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0 \neq f^{(m)}(r)$. Prove that the modified Newton's method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

converges at least quadratically if $x_0$ is sufficiently close to $r$.

**Problem 6**. Let $x_0$ and $x_1$ be two distinct real numbers and let $n$ and $m$ be positive integers.

(a) Find all polynomials $p$ of degree at most $n+m$ such that

$$p(x_0) = p'(x_0) = \cdots = p^{(n-1)}(x_0) = 0,$$
$$p(x_1) = p'(x_1) = \cdots = p^{(m-1)}(x_1) = 0.$$

(b) Let $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_{m-1}$ be any real numbers. Show that there exists one and only one polynomial $q$ of degree at most $n+m-1$ such that

$$q^{(k)}(x_0) = a_k \quad \text{for } k = 0, 1, \ldots, n-1,$$
$$q^{(k)}(x_1) = b_k \quad \text{for } k = 0, 1, \ldots, m-1.$$

(Hint: use induction on $m$.)

**Problem 7.** Let $f$ be a continuous function on $[-1,1]$ that is not identically zero there and suppose

$$\int_{-1}^{1} f(x)p(x)\,dx = 0$$

for all polynomials $p$ of degree at most $n$, where $n$ is a fixed nonnegative integer. Show that $f$ changes sign at least $n+1$ times on $[-1,1]$.
(Hint: Suppose $f$ changes sign only $k$ times where $0 \le k \le n$ and choose $p$ appropriately.)

**Problem 8.** Suppose that $y \in C^3[a,b]$ satisfies

$$y'(t) = f(t, y(t)) \text{ for } t \in [a,b], \quad y(a) = \gamma\,,$$

where $|f(t,v) - f(t,w)| \le L|v-w|$ on $R = [a,b] \times \mathbb{R}$ for some $L > 0$. Given $h = (b-a)/N$ and $t_k = a + kh$, $0 \le k \le N$, consider the one step implicit method

$$y_{k+1} = y_k + h\frac{f(t_k, y_k) + f(t_{k+1}, y_{k+1})}{2}, \quad y_0 = \gamma.$$

(a) Prove that the local truncation error is bounded by $\frac{h^2}{12}M$ where $M = \max_{t\in[a,b]} |y^{(3)}(t)|$. You may use without proof the approximation formula for the trapezoid rule

$$\int_a^b g(t)dt = (b-a)\left(\frac{g(a)+g(b)}{2}\right) - \frac{g''(\xi)}{12}(b-a)^3.$$

(b) Prove that, if $Lh < 1$,

$$|y(t_k) - y_k| \le \frac{\left(\frac{1+Lh/2}{1-Lh/2}\right)^k - 1}{12L} h^2 M \quad \text{for} \quad 0 \le k \le N\,.$$