Preliminary Examination in Numerical Analysis

Jan. 10, 2020

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. If $A = [a_{ij}] \in R^{m\times n}$, show that $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$.

Problem 2. Let A, X and Y be $n \times n$ matrices such that A is invertible, $\|X - A\| \le \epsilon\|A\|$ and $\|Y - A\| \le \epsilon\|A\|$ for some $\epsilon > 0$. If $\epsilon\kappa(A) < 1$, prove that X and Y are invertible and

$$\|X^{-1} - Y^{-1}\| \le \frac{2\epsilon\kappa(A)}{1 - \epsilon\kappa(A)} \min\{\|X^{-1}\|, \|Y^{-1}\|\},$$

where $\kappa(A)$ is the condition number of A.

Problem 3. Let A be a $m \times n$ matrix with $m \ge n$. Suppose that A has full column rank. Show that there is a unique $m \times n$ orthogonal matrix Q ($Q^TQ = I_n$) and a unique $n \times n$ matrix R with positive diagonals $r_{ii} > 0$ such that $A = QR$.

Problem 4. Consider the following iteration for an $m \times n$ matrix A (with $m > n$) given some initial vector $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} y_i &= \frac{Ax_i}{\|Ax_i\|_2} \\ x_{i+1} = &= \frac{A^T y_i}{\|A^T y_i\|_2} \end{aligned}$$

for $i = 0, 1, \cdots$. Under what conditions on the singular values of A and the initial vector x_0 will this iteration converge, and what will x_i and y_i converge to? Provide a proof to your answers.

Problem 5. Assume that $f \in C^3[a, b]$ with $f^{'}(x) > 0$ for all x. Let $r \in [a, b]$ be a root of f satisfying $f(r) = f''(r) = 0$ and let $\{x_n\}$ be generated by Newton's method. If $x_n \in [a, b]$, prove that $|x_{n+1} - r| \le \frac{M}{2m}|x_n - r|^3$, where $M = \max_{x\in[a,b]} |f^{(3)}(x)|$ and $m = \min_{x\in[a,b]} f^{'}(x)$.

Problem 6. It can be verified that $g(x) = (2x - 1)(10x^2 - 10x + 1)$ satisfies $\int_0^1 x^n g(x)\, dx = 0$ for $n = 0, 1, 2$. Find numbers x_0,x_1,x_2 and w_0,w_1,w_2 satisfying

$$\int_0^1 p(x)\, dx = w_0 p(x_0) + w_1 p(x_1) + w_2 p(x_2)$$

for all polynomials p of degree at most 5. Cite the theorems you use.

Problem 7. Let (p, q) be an inner product on the vector space V of all real polynomials of a single variable and suppose $(e_1 p, q) = (p, e_1 q)$ for all polynomials p and q in V, where e_1 is the polynomial $e_1(x) = x$. Define inductively a sequence $\{p_n\}$ of polynomials in V by $p_0(x) = 1$, $p_1(x) = x - a_1$ and

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x),$$

where $a_n = \dfrac{(e_1 p_{n-1}, p_{n-1})}{(p_{n-1}, p_{n-1})}$ for $n \geq 1$, $b_n = \dfrac{(e_1 p_{n-1}, p_{n-2})}{(p_{n-2}, p_{n-2})}$ for $n \geq 2$.

Show that $(p_n, p_m) = 0$ when $n, m \geq 0$ and $m \neq n$.

Problem 8. Prove the following theorem using only elementary calculus: Let $f(x)$ be a function with continuous derivatives in $[a, b]$ up to and including the $n + 1$th derivative and let $p(x)$ be a polynomial of degree at most n with $p(x_i) = f(x_i)$ for $i = 0, 1, \ldots n$, where $a \leq x_0 \leq x_1 \cdots \leq x_n \leq b$. Show that to each x in $[a, b]$ there corresponds a point ξ in (a, b) such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i).$$