Preliminary Examination in Numerical Analysis

Jan 4, 2022

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Let $A \in \mathbb{R}^{n \times n}$ be strictly column diagonally dominant, i.e., $\sum_{i \neq j} |a_{ij}| < |a_{jj}|$ for $j = 1, \ldots, n$. Show that no row interchanges are needed when applying Gaussian elimination with partial pivoting to A.

Problem 2. Let $A \in \mathbb{R}^{m \times n}$ $(m < n)$ be full rank. Show that the solution of the problem $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|_2$ is an $(n-m)$-dimensional set, and then derive its unique minimum norm solution using modified normal equations and QR decomposition.

Problem 3. Consider a real symmetric matrix $H = \begin{bmatrix} A & B \\ B^T & \mathbf{0} \end{bmatrix}$ where $B \in \mathbb{R}^{m \times n}$ has full rank with $m \geq n$. Denote the eigenvalues of a symmetric matrix X by $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$.

(a) If $A = \mathbf{0}$ and $m = n$, then use the singular values of B to express the condition number of H.

(b) Show that $\lambda_j(H) \geq \lambda_j(A)$ for $j = 1, \ldots, m$. (*Hint:* Use Courant-Fischer minimax theorem.)

Problem 4. Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$.

(a) Show the shifted QR iteration algorithm preserves the upper Hessenberg form.

(b) Design a two-phase algorithm using (a) to find the eigenvalues of A and specify how to choose shifts to achieve cubic convergence.

Problem 5. Let $f(x) = \frac{1}{1+9x^2}$.

(a) Use Newton's interpolation formula to obtain the interpolating polynomial $p(x)$ for $f(x)$ using divided differences at the points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

(b) Find an expression for the error bound of $p(x)$ in (a). That is, write out an upper bound for $|\ f(x) - p(x)\ |$.

Problem 6. For each of the following cases:

(a) $f(x) = \sqrt{x}$,

(b) $f(x) = x^2$,

(c) $f(x) = xe^x$,

consider the Newton's method for solving the equation $f(x) = 0$ and describe the convergence behavior. That is, specify whether it diverges, converges linearly or quadratically.

Problem 7. Describe how to determine the nodes t_k and weights w_k for the Gauss quadrature rule

$$\int_0^1 f(t)\omega(t)dt \approx \sum_{k=1}^{n} f(t_k)w_k,$$

where $\omega(t) > 0$ is a given weight function. You can assume $\int_0^1 p(t)q(t)\omega(t)dt$ can be evaluated exactly for all polynomials $p(t)$ and $q(t)$, and the roots of a polynomial can also be computed.

Problem 8. Consider the following two-stage numerical method

$$\boldsymbol{y}_{\text{next}} = \boldsymbol{y} + h\boldsymbol{f}\left(x + \frac{1}{2}h,\ \boldsymbol{y} + \frac{1}{2}h\boldsymbol{f}(x, \boldsymbol{y})\right)$$

for solving the initial value problem $\boldsymbol{y}'(x) = \boldsymbol{f}(x, \boldsymbol{y})$ and $\boldsymbol{y}(0) = \boldsymbol{y}_0$. Find out the order of the truncation error for this method and determine if the method is A-stable. Explain why.