Preliminary Examination in Numerical Analysis

Jan 7, 2026

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Let $x_1, x_2, \cdots, x_n$ be floating point numbers. Use $fl(a+b) = (a+b)(1+\delta)$ (where a, b are any two machine numbers and $|\delta| < \epsilon$) to prove that

$$fl\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} x_i(1+\delta_i),$$

where δ_i is some number satisfying $|\delta_i| \le (n-1)\epsilon + \mathcal{O}(\epsilon^2)$ and ϵ is the machine precision.

Problem 2. Let $\|\cdot\|$ be a matrix norm satisfying $\|XY\| \le \|X\|\|Y\|$. Let X and Y be invertible matrices such that $\|X - Y\| < \frac{1}{\|X^{-1}\|}$. Let $\delta = \|X^{-1}\|\|X - Y\|$.

a) Show that $\|Y^{-1}\| \le \frac{\|X^{-1}\|}{1-\delta}$.

b) Show that $\frac{\|X^{-1}-Y^{-1}\|}{\|X^{-1}\|} \le \frac{\kappa(X)}{1-\delta}\frac{\|X-Y\|}{\|X\|}$, where $\kappa(X)$ is the condition number of X.

Problem 3. Let $A = [a_{ij}] \in \mathbb{R}^{(n+1)\times n}$ be in the upper Hessenberg form, i.e., $a_{i,j} = 0$ for all $i > j+1$. Assume that $a_{i+1,i} \ne 0$ for $1 \le i \le n$. Describe an efficient method (with $O(n^2)$ operations) to solve the least squares problem

$$\min_x \|Ax - b\|_2.$$

Does the least squares problem have a unique solution? Explain. Hint: Use QR decomposition and note that a Hessenberg matrix is almost in the upper triangular form.

Problem 4. Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric positive definite matrix with eigenvalues

$$\lambda_1 > \lambda_2 \ge \cdots \ge \lambda_n,$$

and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$. Consider the power iteration

$$x_{k+1} = \frac{Ax_k}{\|Ax_k\|_2}, \quad \text{for } k = 0, 1, 2, \ldots.$$

Show that, for any initial vector x_0 with a nonzero component in the direction of v_1, the iterates x_k converge to $\pm v_1$, and determine the asymptotic rate of convergence.

Problem 5. Derive the composite trapezoidal rule for numerical integration,

$$\int_a^b f(x)\,dx = h\left(\frac{1}{2}f_0 + f_1 + \cdots + f_{n-1} + \frac{1}{2}f_n\right) - \frac{h^2}{12}(b-a)f''(\xi),$$

where $a < \xi < b$.

Problem 6. Consider the problem of approximating $\sqrt{2}$ as the root of $f(x) = x^2 - 2$.

(a) Describe the secant method for solving this root finding problem.

(b) Show that the method is locally superlinearly convergent.

Problem 7. Describe how to determine the nodes t_k and weights w_k for the Gauss quadrature rule

$$\int_0^1 f(t)\omega(t)dt \approx \sum_{i=1}^{n} f(t_k)w_k,$$

where $\omega(t) > 0$ is a given weight function. You can assume $\int_0^1 p(t)q(t)\omega(t)dt$ can be evaluated exactly for all polynomials $p(t)$ and $q(t)$, and the roots of a polynomial can also be computed.

Problem 8. Describe what is A-stability and show that the following implicit formula is A-stable:

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \frac{1}{2}h\left(f(t_n, \boldsymbol{x}_n) + f(t_{n+1}, \boldsymbol{x}_{n+1})\right).$$