Preliminary Examination in Numerical Analysis

June 1, 2000

Instructions:

- Examination is 3 hours: 9am to noon
- Part I: Numerical Linear Algebra
- Part II: Numerical Differential Equations
- For problems with multiple parts, you may get full credit for a particular part using results of previous parts even though you fail to finish these previous parts
- For Part I: Do **Problem 1**, and one of **Problem 2** and **Problem 3**

Part I. Numerical Linear Algebra

Problem 1. **Rank-revealing QR decomposition**. R is $n \times n$ and upper triangular. $\sigma_{\min}(\cdot)$ denotes the smallest singular value. I is the $n \times n$ identity matrix whose jth column is e_j. P_{ij} is I with its ith and jth columns swapped, i.e.,

$$P_{ij} = \begin{array}{c} \\ \\ \\ i \rightarrow \\ \\ \\ \\ j \rightarrow \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & \overset{i}{\downarrow} & & & & \overset{j}{\downarrow} & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & & & & 1 & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & 1 & & & & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}.$$

1. RP_{ij} is R with its ith and jth columns swapped.
2. Show that if $\|Rx\|_2 = \epsilon$ and $\|x\|_2 = 1$ then $\sigma_{\min}(R) \le \epsilon$.
3. It is reasonable to expect that R has a small diagonal entry if R is nearly singular. Nevertheless this is not true. Consider

   $$R = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ & 1 & -1 & \cdots & -1 \\ & & \ddots & \ddots & -1 \\ & & & \ddots & -1 \\ & & & & 1 \end{pmatrix},$$

   which has no small diagonal entries, but $\sigma_{\min}(R) \le 2^{-n+1}\sqrt{n}$. Prove this.
   Hint: calculate Rv *where* $v = (2^{n-1}, 2^{n-2}, \ldots, 2, 1)^T$.
4. Let vector x, $\|x\|_2 = 1$ satisfy $\|Rx\|_2 = \epsilon$, and j_0 be the integer such that $\|x\|_\infty = |x_{j_0}|$.

(a) Show that $|x_{j_0}| \geq n^{-1/2}$.

(b) Present an algorithm to compute a QR decomposition of $RP_{j_0 n}$

$$RP_{j_0 n} = \widetilde{Q}\widetilde{R}.$$

at the cost of $\mathcal{O}(n^2)$ flops.

(c) Show that the last entry $\widetilde{r}_{nn}$of $\widetilde{R}$ satisfies $|\widetilde{r}_{nn}| \leq \sqrt{n}\,\epsilon$, and thus $\widetilde{R}$ must have a small diagonal entry if ϵ is small.

Problem 2. Let A be $n \times m$ and $n > m$.

1. What is Gram-Schmidt process to orthogonalize the columns of A? Formulate the process into matrix factorization $A = QR$, and what is Q and what is R? Does Gram-Schmidt process always produce vectors orthogonal up to around machine epsilon?

2. Describe a way that always produce fully orthogonal vectors from the columns of A? "*Fully orthogonal*" means orthogonal up to around machine epsilon.

3. Compare the speed of Gram-Schmidt process and the method you just described.

Problem 3. Let $A = D + \rho uu^T$, where $D = \text{diag}\,(d_1, d_2, \ldots, d_n)$ and $u = (u_1, u_2, \ldots, u_n)^T$. All numbers are real.

1. Show that if $u_i = 0$, then d_i is an eigenvalue of A and the corresponding eigenvector is e_i, the ith column of the identity matrix.

2. Show that if $d_i = d_{i+1}$, then d_i is an eigenvalue of A. Derive an expression for the corresponding eigenvector.

3. Assume that all $u_i \neq 0$ and that $d_1 < d_2 < \cdots < d_n$ and $\rho > 0$.

   (a) Show that the eigenvalues of A are the roots of

$$1 + \rho \sum_{j=1}^{n} \frac{u_i^2}{d_j - \lambda} = 0.$$

(b) Show that this equation has n roots and find n open intervals each of which contains exactly one root.

(c) Show that if λ is a root then the corresponding eigenvector is parallel to $(D - \lambda I)^{-1}u$.

Part II. Numerical Differential Equations

Problem 4. This problem is concerned with *stability* and *boundedness* analysis for an explicit scheme for the following convection-diffusion problem

$$u_t + a u_x - \varepsilon u_{xx} = 0, \tag{1}$$

where a and ε are convection and diffusion coefficients, respectively, and an initial data $u(x,0) = u_0(x) \geq 0$ is assumed.

(a) Formulate the forward-time central-space finite difference scheme for (1), with a uniform space step h and time step k.

(b) Prove that the scheme is unconditionally unstable when $\varepsilon = 0$. (You may utilize the von Neumann analysis.)

(c) Find the stability condition when $a = 0$.

(d) Given a and ε, find conditions for the numerical solution to be nonnegative.

Problem 5. Let $\Omega = (0,1)^2$ and its boundary $\Gamma = \partial\Omega$. Consider the boundary value problem: find $u \in H_0^1(\Omega)$ such that

$$\begin{array}{lll} \text{(i)} & -\Delta u = f(x,y), & (x,y) \in \Omega, \\ \text{(ii)} & u = 0, & (x,y) \in \Gamma, \end{array} \tag{2}$$

where Δ is the Laplace operator and f is the source function.

(a) Derive the weak form of (2). Sketch an argument which shows that (2) and its weak form have the same solutions, provided that u is sufficiently smooth.

(b) Let $\mathcal{T}_h$ be a uniform triangulation of Ω, where $h = 1/(N+1)$ for some positive integer N, V^h the space of piecewise linear functions defined on $\mathcal{T}_h$, and $u^h \in V^h$ the Galerkin approximation of u. Let $\|\cdot\|$ denote the $L^2(\Omega)$-norm. Then, one can see

$$\|\nabla(u - u^h)\| \leq \|\nabla(u - v)\|, \quad \text{for any } v \in V^h. \tag{3}$$

Prove (3) and describe, in detail, its implications in error analysis for the finite element solution.

Problem 6. Consider the initial-boundary value problem

$$\begin{aligned} u_t - u_{xx} - u_{yy} &= f(x,y,t), && (x,y,t) \in \Omega \times J, \\ u(x,y,t) &= 0, && (x,y,t) \in \Gamma \times J, \\ u(x,y,0) &= u_0(x,y), && (x,y) \in \Omega, \end{aligned} \tag{4}$$

where $\Omega = (0,1)^2$, $\Gamma = \partial\Omega$, and $J = (0,T]$ for some $T > 0$.

(a) Formulate the Crank-Nicolson central finite difference scheme.

(b) Indicate the accuracy order of the scheme.

(c) Formulate the alternating direction implicit (ADI) method by perturbing the formulation obtained in (a); discuss its efficiency and pitfalls in accuracy.