Preliminary Examination in Numerical Analysis

May 31, 2019

Instructions:

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

Problem 1. Let $fl\,(e)$ denote the computational result of an expression e in a floating point arithmetic and let ϵ be the machine roundoff unit. Consider computing a summation $S = \sum_{i=1}^{n} x_i$ for n machine numbers $x_1, x_2, \cdots, x_n$. Prove that $fl\,(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} x_i(1 + \delta_i)$ with $\delta_i \leq (n-1)\epsilon + \mathcal{O}(\epsilon^2)$. Under what condition on x_i can S be computed with a relative error in the order of ϵ?

Problem 2. Let $A \in \mathbb{R}^{m \times n}$ be a *rectangular* matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and $b \in \mathbb{R}^m$. Assume $Ax = b$ has a solution x. Let $\hat{x}$ be an approximate solution and let $r = b - A\hat{x}$. Prove that $A(x - \hat{x}) = r$ and

$$\frac{\|x - \hat{x}\|_2}{\|x\|_2} \leq \kappa(A) \frac{\|r\|_2}{\|b\|_2},$$

where $\kappa(A) = \|A\|_2 \|A^{\dagger}\|_2 = \frac{\sigma_1}{\sigma_n}$.
(Hint: use SVD of A.)

Problem 3. Let $A \in \mathbb{C}^{n \times n}$. If λ is an eigenvalue of A, show that there exists an index i such that

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

Problem 4.
(a) Describe for a general 4×4 matrix A how to compute the QR factorization of A using the Householder reflection.
(b) Describe for a general 4×4 symmetric matrix A how to compute an orthogonal matrix V_0, such that $V_0^T A V_0$ is a tridiagonal matrix.
For both parts, clearly outline the precise transformations used and the resulting matrices for each step.

Problem 5. Let r be a root of $f \in C^2(R)$. Assume that $f'(x) > 0$ and $f''(x) > 0$ for all $x > r$. Prove that for any $x_0 > r$, Newton's method converges to r.

Problem 6. Let $-1 \le x_0 < x_1 < \cdots < x_n \le 1$ be distinct numbers and let

$$A_n := \max\{|(x - x_0)(x - x_1) \cdots (x - x_n)| : \ -1 \le x \le 1\}.$$

a) Given n, how should $x_0, x_1, \ldots, x_n$ be chosen so that A_n is as small as possible. (Hint: The x_i's should not be uniformly spaced.)

b) Give the value of A_n for the choice of $x_0, x_1, \ldots, x_n$ in part (a).

Problem 7. Let $p_0(x), p_1(x), \ldots, p_n(x)$ be polynomials, where $p_0(x) = 1$ and $p_k(x)$ has degree k for $k = 1, \ldots, n$. Suppose $w(x)$ is a positive continuous function on $[-1, 1]$ satisfying

$$\int_{-1}^{1} p_i(x)^2 w(x)\, dx = 1, \quad \int_{-1}^{1} p_i(x) p_j(x) w(x)\, dx = 0$$

for any $0 \le i, j \le n$ with $j \ne i$.

a) Show that the minimum value of $\displaystyle\int_{-1}^{1} p(x)^2 w(x)\, dx$ over all monic polynomials $p(x)$ of degree n is $1/C_n^2$, where C_n is the coefficient of x^n in $p_n(x)$. (Recall that a monic polynomial is one where the coefficient of the highest power is 1.)

b) Find all polynomials $p(x)$ in part (a) where the minimum is attained and justify your answer.

Problem 8. We wish to solve initial value problems of the form $x'(t) = f(t, x)$, $x(t_0) = x_0$. Let h be the step size and let $t_i = t_0 + ih$ for $i = 0, 1, \ldots$. Compute the coefficients A, B and C in a multistep method of the form

$$x(t_{n+1}) = x(t_n) + h[Af(t_n, x(t_n)) + Bf(t_{n-1}, x(t_{n-1})) + Cf(t_{n-2}, x(t_{n-2}))].$$

Your formula should be accurate when $f(t, x) = at^2 + bt + c$.