# Preliminary Examination in Numerical Analysis

June 4, 2025

**Instructions:**

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

**Problem 1**. Let $U$ be an $n \times n$ nonsingular upper triangular matrix. Write down the backward substitution algorithm for solving $Ux = b$ where $b \in \mathbf{R}^n$. Let $\hat{x}$ be the computed solution to $Ux = b$ using this algorithm. Prove that $(U + \delta U)\hat{x} = b$ for some $\delta U$ with $|\delta U| \leq n\epsilon |U|$
(You may use without proof the fact that $fl(\sum_{i=1}^{d} x_i y_i) = \sum_{i=1}^{d} x_i y_i (1 + \delta_i)$ with $|\delta_i| \leq d\epsilon$, ignoring all higher order terms in $\epsilon$.)

**Problem 2**. Let $A$ be an $n \times n$ invertible matrix and $U$ and $V$ be $n \times k$ (with $n \geq k$) matrices. Prove that the following are equivalent:

(a) $\begin{pmatrix} A & U \\ V^T & I \end{pmatrix}$ is invertible;

(b) $I - V^T A^{-1} U$ is invertible;

(c) $A - UV^T$ is invertible.

**Problem 3**. Let $U$ and $V$ be two real $n \times n$ orthogonal matrices and $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ with $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n$. If $U\Sigma = \Sigma V$, prove that $U$ and $V$ are diagonal matrices of $\pm 1$.

**Problem 4**. Let $A$ and $B$ be $n \times n$ symmetric matrices, and suppose $B$ is positive definite. Show that

$$\lambda_k(A + B) \geq \lambda_k(A),$$

for $k = 1, 2, \ldots, n$, where $\lambda_k(\cdot)$ denotes the $k$-th largest eigenvalue of a matrix. Hint: use Courant-Fischer min-max theorem.

**Problem 5**. Consider $f(x) = \dfrac{1}{1 + 25x^2}$ with nodes $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$.

(a) Construct the divided difference table and write the Newton's interpolating polynomial $p_2(f; .)$ corresponding to these nodes.

(b) Show that

$$\|f - p_2(f; .)\|_\infty \leq \frac{M_3}{9\sqrt{3}}, \text{ where } M_3 = \|f'''\|_\infty.$$

**Problem 6**. Consider the problem of approximating $\sqrt{2}$ as the root of $f(x) = x^2 - 2$.

(a) Describe Newton's iterative procedure for solving this root finding problem.

(b) Prove that the convergence of the algorithm in part (a) is quadratic with an appropriate choice of initial value $x_0$.

**Problem 7**.

(a) Construct the weighted Newton-Cotes formula

$$\int_0^1 f(x)x^\alpha dx = a_0 f(0) + a_1 f(1) + E(f), \quad \alpha > -1.$$

(b) Derive an expression for the error term $E(f)$ in terms of a derivative of $f$.

(c) From the formulae in (a) and (b) derive an approximate integration formula for $\int_0^h f(t)t^\alpha dt$ ($h > 0$ small), including an expression for the error term.

**Problem 8**. Consider the multi-step method for the IVP $y' = f(x, y)$ with $x \in [a, b]$, $y(a) = y_0$. Describe the Predictor-Corrector scheme using the two-step Adams-Bashforth and the two-step Adams-Moulton methods.