# Preliminary Examination in Numerical Analysis

June 5, 2026

**Instructions:**

1. The examination is for 3 hours.
2. The examination consists of eight equally-weighted problems.
3. Attempt all problems.

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**Problem 1**. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Show that Gaussian elimination can be performed successfully on $A$ without any row interchanges.

**Problem 2**. Let $P \in \mathbb{R}^{n\times n}$ be a projection matrix, i.e., $P^2 = P$. Is it possible for $\|P\|_2 = 2$? If so, provide an example. Otherwise, prove that no such projection matrix exists.

**Problem 3**. Given $A \in \mathbb{R}^{m\times n}$ and $r \le \min\{m, n\}$, determine a best rank-$r$ approximation of $A$ in the matrix 2-norm $\|\cdot\|_2$ using the SVD of $A$. A rigorous justification of your solution is required.

**Problem 4**. Let $A \in \mathbb{C}^{n\times n}$. For $\varepsilon > 0$ and $z \in \mathbb{C}$, show that

$$z \text{ is an eigenvalue of } A + \Delta A \text{ for some } \Delta A \in \mathbb{C}^{n\times n} \text{ with } \|\Delta A\|_2 \le \varepsilon,$$

if and only if

$$\|(zI - A)^{-1}\|_2 \ge \varepsilon^{-1},$$

with the convention that $\|(zI - A)^{-1}\|_2 = \infty$ when $z$ is an eigenvalue of $A$.

**Problem 5**. Find an expression for the interpolation error (no proof required) for the function $f(x) = x^2$ using a linear interpolation $p_1(x)$ at the nodes $x_0, x_1 \in [-2, 2]$. Then determine the values of $x_0, x_1$ that minimize the maximum error on the interval $[-2, 2]$:

$$\max_{x\in[-2,2]} |f(x) - p_1(x)|.$$

**Problem 6**. Find the two-point Gauss quadrature rule

$$\int_{-1}^{1} f(x)x^4 dx \approx a_0 f(x_0) + a_1 f(x_1).$$

**Problem 7**. Suppose $f(x)$ has a root $\alpha$ such that $f(\alpha) = f'(\alpha) = 0$ and $f''(\alpha) \ne 0$. Is the Newton method locally convergent to $\alpha$? If so, determine the rate of convergence.

**Problem 8**. Consider the following linear multi-step method for solving the initial value problem $dx/dt = f(t, x)$ with $x(0) = x_0$:

$$x_{i+2} = 4x_{i+1} - 3x_i - 2hf(t_i, x_i), \quad i = 0, \dots, N-2,$$

where $h$ is the step size and $t_i = ih$. Analyze the order, stability, and convergence of this method.