Preliminary Examination on Partial Differential Equations

January 8, 2001

Instructions

This is a three-hour examination. You are to work a total of **five problems**. Please indicate clearly on your test paper which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Problem 1. Let Ω be a bounded smooth domain in $\mathbb{R}^n$.

(a). Assume that $u \in C^2(\Omega)$ is a harmonic function. State and prove the mean value property of u.

(b). Let $v \in C(\overline{\Omega})$. Suppose that, for any $x \in \Omega$, there exists $r_x > 0$ such that $B(x, r_x) \subset \Omega$ and

$$v(x) = \frac{1}{|B(x,r_x)|} \int_{B(x,r_x)} v(y)\,dy.$$

Prove that v cannot achieve its maximum in the interior of Ω, unless v is constant.

Problem 2. If $f \in L^1(\mathbb{R})$, let

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} f(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy$$

where $x \in \mathbb{R}$ and $t > 0$.

(a). Give a careful proof that the partial derivative

$$\frac{\partial u}{\partial x}$$

exists.

(b). Show that there exists a constant C (which should depend on f) so that for all $x \in \mathbb{R}$ and $t > 0$,

$$\left| \frac{\partial u}{\partial x}(x,t) \right| \le \frac{C}{t}.$$

Problem 3. Let Ω be a smooth domain in $\mathbb{R}^n$ and $\Omega_T = \Omega \times (0, T]$ for $T > 0$. Suppose that $u \in C^2(\overline{\Omega_T})$ solves

$$u_{tt} - \Delta u = 0$$

in Ω_T. Fix $(x_0, t_0) \in \Omega_T$. Consider the cone

$$C = \{(x,t) : \ \ 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}.$$

Show that, if $C \subset \overline{\Omega_T}$ and $u(x,0) = u_t(x,0) = 0$ on $B(x_0, t_0)$, then $u \equiv 0$ within the cone C.

Problem 4. Use the method of characteristic to find a solution of the initial value problem:

$$\begin{cases} u_x + u_y = u^2 & \text{near y=0} \\ u(x,0) = x^2 & x \in \mathbb{R}. \end{cases}$$

Problem 5. Let Ω be a bounded smooth domain in $\mathbb{R}^n$.

(a). State the definition of the Sobolev space $W^{1,p}(\Omega)$.

(b). Let $u \in W^{1,p}(\Omega)$ for some $1 \le p < \infty$. Show that there exists a sequence of functions u_j in $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that

$$u_j \to u \quad \text{in } W^{1,p}(\Omega),$$

as $j \to \infty$.

Problem 6. Let Ω be a bounded smooth domain in $\mathbb{R}^n$, and $A(x) = (a_{jk}(x))_{1 \le j,k \le n}$ where a_{jk} are bounded measurable functions on Ω. Define

$$L = \sum_{j,k} \frac{\partial}{\partial x_j}(a_{jk}(x)\frac{\partial}{\partial x_k}).$$

Suppose $a_{jk} = a_{kj}$ and

$$\lambda|\xi|^2 \le \sum_{j,k} a_{jk}\xi_j\xi_k \le \Lambda|\xi|^2, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$$

for some $0 < \lambda < \Lambda < \infty$.

(a). For a given $f \in L^2(\Omega)$. State the definition for a function $u \in H_0^1(\Omega)$ to be a weak solution of the boundary value problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that there exists one and only one weak solution to this boundary value problem.

(b). Prove that any weak solution to the boundary value problem in part (a) satisfies the estimate

$$\int_\Omega |Du|^2\,dx \le C \int_\Omega |f|^2\,dx,$$

where $C > 0$ depends only on A and Ω.

Problem 7. Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Suppose $\psi \in C(\overline{\Omega}) \cap H^1(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of

$$Lu = 0 \quad \text{in } \Omega$$
$$u - \psi \in H_0^1(\Omega).$$

where the elliptic operator L is given in problem 6. Show that

$$u(x) \le \max_{\partial\Omega} \psi$$

for a.e. $x \in \Omega$.