Preliminary Examination in Partial Differential Equations
January 11, 2010

Instructions This is a three-hour examination. The exam is divided into two parts. You need to solve a total of five problems. You must do at least two problems from each part. Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART I

(1) Suppose that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves the equation

$$(\Delta u)(x) + \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} + c(x) u = 0$$

where $c(x) < 0$ in $\Omega \subset \mathbf{R}^n$. Show that $u = 0$ on $\partial\Omega$ implies that $u = 0$ in Ω. *Hint*: show that $\max u \leq 0$ and $\min u \geq 0$.

(2) The fundamental solution to the heat equation,

$$\Delta u(x,t) = \frac{\partial u}{\partial t}(x,t),$$

relative to (0,0) in $\mathbf{R}^{n+1}$ is given by

$$\Gamma(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right) & x \in \mathbf{R}^n, t > 0, \\ 0 \text{ when } x \in \mathbf{R}^n, t \leq 0. \end{cases}$$

Γ has the properties that

$$\int_{\mathbf{R}^n} \Gamma(x,t)\, dx = 1 \text{ when } t > 0$$

$$\lim_{t \downarrow 0} \int_{|x|>\delta} \Gamma(x,t)\, dt = 0 \text{ whenever } \delta > 0.$$

Show that if $f \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the function

$$v(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y,t) f(y)\, dy, (x,t) \in \mathbf{R}^n \times (0,\infty),$$

obeys

$$\lim_{\substack{(x,t)\to(x^0,0) \\ x\in\mathbf{R}^n,\ t>0}} v(x,t) = f(x^0) \text{ whenever } x^0 \in \mathbf{R}^n.$$

(3) Suppose that Ω is a bounded open subset of $\mathbf{R}^3$ with smooth boundary and let u be a $C^2(\bar{\Omega} \times [0,T)$ solution of the initial-boundary value problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = u_t(x,t) & (x,t) \in \Omega \times [0,T) \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T) \end{cases}$$

Define the energy of a solution u by

$$E(t) = \frac{1}{2} \int_\Omega u_t(x,t)^2 + |\nabla u(x,t)|^2 \, dx.$$

Show that

$$E(t) \leq \exp(2t) E(0), \qquad t \in [0,T).$$

(4) Find a solution of the initial-value problem

$$\begin{cases} u_t + u u_x = 0, & t > 0, \; x \in \mathbf{R} \\ u(x,0) = x^2, & x \in \mathbf{R} \end{cases}$$

Explain why there is only one smooth solution of this initial-value problem.

PART II

(5) Let $u \in W^{1,p}(\mathbf{R}^n)$ for some $p, 1 \leq p < \infty$. Let $B(x,r) = \{y \in \mathbf{R}^n : |y - x| < r\}$ and let $\phi \in C_0^\infty(B(0,1))$ with $\int_{\mathbf{R}^n} \phi dx = 1$. Set $\phi_\epsilon(x) = \epsilon^{-n} \phi_\epsilon(x)$ for $x \in \mathbf{R}^n$ and $\epsilon > 0$. Put

$$u_\epsilon(x) = \phi_\epsilon * u(x) = \int_{\mathbf{R}^n} \phi(x-y) \, u(y) dy, \; x \in \mathbf{R}^n.$$

(a) Use the Lebesgue differentiation theorem and/or Hölder's inequality to show that $u_\epsilon \to u$ pointwise almost everywhere and in the norm of $W^{1,p}(\mathbf{R}^n)$.

(b) Use (a) to show that if $\eta \in C_0^\infty(\mathbf{R}^n)$, then $u\eta \in W^{1,p}(\mathbf{R}^n)$.

(c) Use (a), (b) to show that there exists a sequence $\zeta_j, j = 1, 2, \ldots$ of infinitely differentiable fuctions with compact support and $\|u - \zeta_j\|_{W^{1,p}(\mathbf{R}^n)} \to 0$ as $j \to \infty$.

(6) Let $u \in C^1(\mathbf{R}^n)$ and let $B(x,r)$ be as in problem 5.

(a) Prove that

$$(*)\ |u(y) - u_{B(x,r)}| \leq c \int_{B(x,r)} \frac{|Du|(z)}{|y-z|^{n-1}}\, dz,\ y \in B(x,r),$$

where $c = c(n)$, and $u_{B(x,r)}$ is the average of u on $B(x,r)$.

(b) If $p > n$ and $v \in W^{1,p}(\mathbf{R}^n)$, use $(*)$ to prove Morrey's theorem: For almost every $x, y \in \mathbf{R}^n$,

$$|v(x) - v(y)| \leq c'|x-y|^{1-n/p}\, \|Dv\|_{L^p(\mathbf{R}^n)},$$

where $c' = c'(p,n)$. You may assume the conclusion of 5 (c).

(7) Let $u \in W^{1,2}(\mathbf{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u(x) + b(u(x)) = f(x), x \in \mathbf{R}^n,$$

where $f \in L^2(\mathbf{R}^n)$ and $b \in C^1(\mathbf{R})$ with $b(0) = 0$, while $b' \geq 0$ on $\mathbf{R}$. Use the method of difference quotients to show that $u \in W^{2,2}(\mathbf{R}^n)$. You may assume that $Du \in L^2(\mathbf{R}^n)$ if and only if $D_k^h u \in L^2(\mathbf{R}^n)$ for $0 < h \leq 1$, where $D_k^h u(x) = \frac{u(x+he_k)-u(x)}{h}$, $x \in \mathbf{R}^n$, while $e_k, 1 \leq k \leq n$, is the point with 1 in the k th coordinate position and 0's elsewhere.

(8) Let $u \in C^2(B(0,1)) \cap C^1(\bar{B}(0,1))$ where $B(0,1)$ is defined as in problem 5 and $\bar{B}(0,1)$ is the closure of this set. Suppose also that $u \geq 0$ is a classical solution to

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j}(x) = 0 \text{ in } B(0,1) \text{ with } u \equiv 0 \text{ on } \partial B(0,1).$$

In this display $(a_{ij}(x))$ are continuous in $\bar{B}(0,1)$ and

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, x \in \bar{B}(0,1),$$

for some $0 < \lambda, \Lambda < \infty$, and all $\xi \in \mathbf{R}^n$. If $m = \min_{B(0,1/2)} u$ show that

$$|Du(x)| \geq m/c \text{ for } x \in \partial B(0,1),$$

where c depends only on n, λ, Λ.